2.5 hrs → Final: Tue Dec 17th
Review session (Prob. Mon 16th
Look at Past Exams

14. The Euler Scheme (4/12/2024)

WW14 not for credit.

Goals.

(1) Review ODE; *derive* an ODE
(2) Solving an ODE numerically
  (a) The power of linear approximation
  (b) Calculating by hand
  (c) Calculating on a computer

Last Time. Diff Eqns

Equations: (1) unknown is a function
(2) involves derivatives of the unknown

Focus: Notion of solution, verifying analytic solutions (inc. idea of Ansatz)
(particular solutions from initial condition)

Last time: analytic exact solutions (with formulas)

Today: numerical solutions (approximate)

Math 100A – WORKSHEET 14
EULER'S METHOD

1. Compound interest (Bernoulli 1683)

(1) Suppose you have a \$100 bank balance which earns an annual interest rate of 30%.

(a) Suppose the interest is paid once, at the end of the year. How much would your balance be at that time?

Interest is \$30 = 30% · \$100

Balance is \$100 + 30% \$100 = 130

(b) Suppose instead that interest is paid four times a year. What is the quarterly interest *rate*? What would the balance be at the end of the first quarter? quarterly rate is 7.5 %

balance: \$100 + 7.5% · \$100 = 107.5

(here $\$100\left(1+\frac{30\%}{4}\right)$)

Date: 4/12/2024, Worksheet by Lior Silberman.

(2) Suppose interest is compounded *continuously* and that at a particular time ~~y~~ t the balance is $y(t)$ dollars, where t is measured in years.

(a) What is the approximate interest rate for the period between times $t, t+h$ if h is very small?

$[t, t+h]$ is fraction $\frac{h}{1\text{ year}}$ of the year.

so rate is $h \cdot 30\%$.

(b) What is the balance at time $t+h$?

$$y(t+h) \approx y(t) + y(t) \cdot (h \cdot 30\%)$$
$$= y(t) + (0.3\, y(t))\, h$$

$\Rightarrow \; y'(t) = 0.3 \cdot y(t)$, or $y' = 0.3\, y$

We derived this DE from the model

(c) Suppose further that interest is *compounded*: after every quarter the interest is added to the balance. What would be the balance at the end of the year?

After 2 quarters have: $107.5 + 7.5\% \cdot 107.5$

$$= 107.5 \cdot (1 + 7.5\%)$$

$$= 100 \cdot (1 + 7.5\%)^2$$

so after 4 quarters get

$$100\,(1+0.075)^4 = 100 \cdot \left(1 + \frac{30\%}{4}\right)^4$$

(d) Suppose instead that interest is compounded *daily* and that at a particular day the balance is y dollars. What is the balance the next day?

rate is now $\frac{0.3}{365} = \frac{1\text{day}}{365\text{ days}} \cdot 0.3$

$$= \frac{1/365\text{ year}}{1\text{ year}} \cdot 0.3$$

New balance: $y + \frac{0.3}{365} \cdot y$

$$\left(= \left(1 + \frac{0.3}{365}\right) \cdot y\right)$$

① Use DE to get $y'(0)$.

2. The Euler scheme

(3) Consider the ODE $y' = 0.3y$ from the previous page. We will work on the interval $[0, 1]$ with $y(0) = 100$.

(a) $[n = 1]$ What is $y'(0)$? Approximate $y(1)$ using a linear approximation.

Know $y'(0) = 0.3 \cdot y(0) = 0.3 \cdot 100 = 30$

$y(1) \approx y(0) + y'(0)(1-0) = 100 + 30 \cdot 1 = 130$

(b) $[n = 2]$ Approximate $y\left(\frac{1}{2}\right)$ using a linear approximation. What is $y'\left(\frac{1}{2}\right)$ approximately? Use this to estaimte $y(1)$.

$y\left(\frac{1}{2}\right) \approx y(0) + y'(0) \cdot \frac{1}{2} = 100 + 30 \cdot \frac{1}{2} = 115$

② plug in approx values

Now $y'\left(\frac{1}{2}\right) = 0.3\, y\left(\frac{1}{2}\right) \approx 0.3 \cdot 115 = 34.5$

① DE

$y(1) \approx y\left(\frac{1}{2}\right) + y'\left(\frac{1}{2}\right) \frac{1}{2} \approx 115 + 34.5 \cdot \frac{1}{2} = 132.25$

linear approx

(c) $[n = 3]$ do the same but dividing the interval into three steps.

$y\left(\frac{1}{3}\right) \approx y(0) + y(0) \cdot \frac{1}{3} = 110$ so $y'\left(\frac{1}{3}\right) \approx 0.3 \cdot 110 = 33$

so $y\left(\frac{2}{3}\right) \approx y\left(\frac{1}{3}\right) + 33 \cdot \frac{1}{3} \approx 121$ so $y'\left(\frac{2}{3}\right) \approx 0.3 \cdot 121 = 36.3$

so $y(1) \approx y\left(\frac{2}{3}\right) + y'\left(\frac{2}{3}\right) \cdot \frac{1}{3} \approx 121 + 36.3 \cdot \frac{1}{3} = 133.1$

What do we mean by $y' = f(y; x)$?

Ex $\quad y' = ry, \quad f(y;x) = ry$

Ex $\quad y' = x^3 - xy, \quad f(y;x) = x^3 - xy$

Ex $\quad y' = xy \quad f(y;x) = xy$

The Euler Scheme

Input: ODE $y' = f(y; x)$, $[a,b]$, initial value y_0

(0) choose <u>number</u> n of steps $\Leftrightarrow$ <u>step size</u> $\Delta x = h = \frac{b-a}{n}$

(1) divide $[a,b]$ into steps:

$$x_0 = a,\quad x_1 = a + \Delta x,\quad x_2 = a + 2\Delta x, \ldots, x_n = a + n\Delta x = b$$

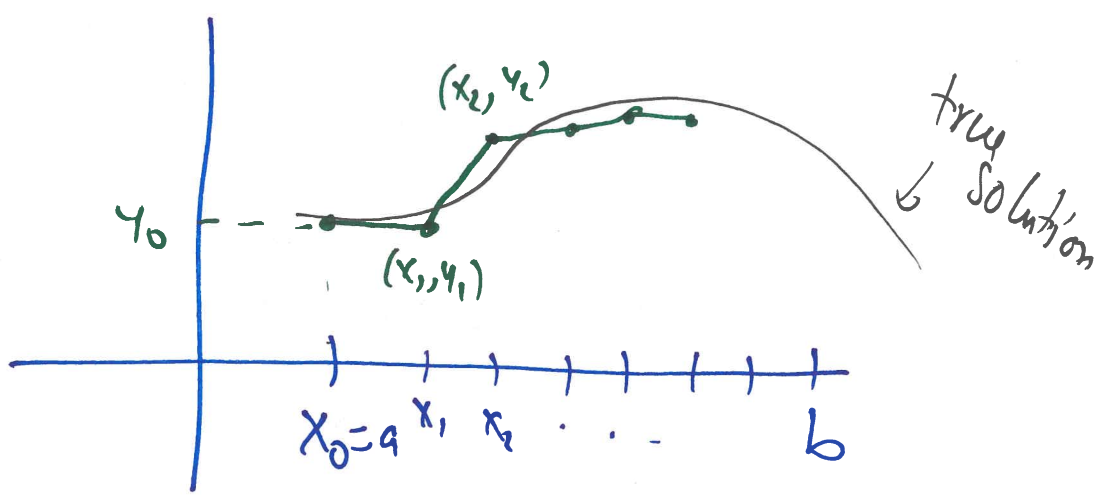
(2) <u>Approximate</u>:

$y_1 = y_0 + f(y_0; x_0)\cdot h \leftarrow$ approx for $y(x_1)$

$y_2 = y_1 + f(y_1; x_1)\, h \leftarrow$ " " $y(x_2)$

$\vdots$

$$\boxed{y_{i+1} = y_i + f(y_i; x_i)\cdot h}$$



(4) Consider the ODE $y' = x^3 - xy$ on the interval $[1, 3]$.

(a) Use two steps of the Euler scheme to approximate $y(3)$ if $y(1) = 0$.

Here $\Delta x = \frac{3-1}{2} = 1$, $x_0 = 1$, $x_1 = 2$, $x_2 = 3$

$y_0 = 0$ ← approx for $y(x_0) = y(1)$

$y_1 = y_0 + f(y_0; x_0)h = 0 + (1^3 - 1 \cdot 0) \cdot 1 = 1$ ← approx for $y(2)$

$y_2 = y_1 + f(y_1; x_1)h = 1 + (2^3 - 2 \cdot 1) \cdot 1 = 7$ ← approx for $y(3)$

So $\boxed{y(3) \approx 7}$

(b) For which A, B, C do we have that $y(x) = Ax^2 + B + Ce^{-x^2/2}$ satisfies the equation?