# Math 100A – SOLUTIONS TO WORKSHEET 1
# EXPRESSIONS AND ASYMPTOTICS

## 1. The ladder of functions

(1) Classify the following functions into *power laws / power functions* and *exponentials*: $x^3$, $\pi x^{102}$, $e^{2x}$, $c\sqrt{x}$, $-\frac{8}{x}$, $7^x$, $8 \cdot 2^x$, $-\frac{1}{\sqrt{3}} \cdot \frac{1}{2^x}$, $\frac{9}{x^{7/2}}$, $x^e$, $\pi^x$, $\frac{A}{x^b}$.

**Solution:** Power laws: $x^3$, $\pi x^{102}$, $c\sqrt{x} = cx^{-1/2}$, $-\frac{8}{x} = -8x^{-1}$, $\frac{9}{x^{7/2}} = 9x^{-7/2}$, $x^e$, $\frac{A}{x^b} = Ax^{-b}$
Exponentials: $e^{2x} = (e^2)^x$, $7^x$, $8 \cdot 2^x$, $-\frac{1}{\sqrt{3}} \cdot \frac{1}{2^x} = -\frac{1}{\sqrt{3}} 2^{-x}$, $\pi^x$.

(2) Order the following functions from small to large *asymptotically as $x \to \infty$:*

(a) $1$, $\sqrt{x}$, $x^{-1/2}$,$x^{1/3}$, $e^x$, $x^{-1/3}$, $10^6 x^{2024}$, $e^{-x}$, $e^{x^2}$, $\frac{2024}{x^{100}}$, $5^x$, $x$.
**Solution:** As $x \to \infty$ we have

$$e^{-x} \ll \frac{2024}{x^{100}} \ll x^{-1/2} \ll x^{-1/3} \ll 1 \ll x^{1/3} \ll x^{1/2} \ll x \ll 10^6 x^{2024} \ll e^x \ll 5^x \ll e^{x^2}$$

(b) Extra: add in $\log x$, $e^{\sqrt{x}}$, $(\log x)^2$, $\log\log x$, $\frac{1}{\log x}$.
**Solution:** As $x \to \infty$ we have

$$e^{-x} \ll \frac{2024}{x^{100}} \ll x^{-1/2} \ll x^{-1/3} \ll \frac{1}{\log x} \ll 1 \ll \log\log x \ll \log x \ll (\log x)^2 \ll x^{1/3} \ll x^{1/2} \ll x \ll 10^6 x^{2024} \ll e^{\sqrt{x}} \ll e^x \ll$$

(c) Repeat (a), this time as $x \to 0^+$.
**Solution:** As $x \to 0$ we have

$$10^6 x^{2024} \ll x \ll x^{1/2} \ll x^{1/3} \ll 1 \sim e^x \sim e^{-x} \sim e^{x^2} \sim 5^x \ll x^{-1/3} \ll x^{-1/2} \ll \frac{2024}{x^{100}}$$

## 2. Asymptotics: simple expressions

(3) How does the each expression behave when $x$ is large? small? what is $x$ is large but negative? Sketch a plot

(a) $1 - x^2 + x^4$ ("Mexican hat potential")
**Solution:** When $x$ is large (positive or negative), $x^4 \gg x^2 \gg 7$ so $7 + x^2 + x^4 \sim x^4$ while when $x$ is small, $7 \gg x^2 \gg x^4$ so $7 + x^2 + x^4 \sim 7$.

(b) $ax^3 - bx^5$ $(a, b > 0)$
**Solution:** When $x$ is very large, $x^5$ dominates $x^3$ so $ax^3 - bx^5 \sim -ax^5$ (which is negative for $x$ positive, positive for $x$ negative!). When $x$ is very small (close to zero), $x^3$ dominates (is bigger than $x^5$ though both are very small) and $ax^3 - bx^5 \sim ax^3$.

(c) $e^x - x^4$
**Solution:** When $x \to \infty$ is very large, $e^x \gg x^4$so $e^x - x^4 \sim e^x$. Near we have $e^x \sim 1 \gg x^4$, so $e^x - x^4 \sim 1$. Finally when $x$ is large but negative $(x \to -\infty)$ we have that $e^x$ decays while $x^4$ grows, so $e^x \ll x^4$ and $e^x - x^4 \sim -x^4$.

(d) Wages in some country grow at 2% a year (so the wage of a typical worker has the form $A \cdot (1.02)^t$ where $t$ is measured in years and $A$ is the wage today). The cost of healthcare grows at 4% a year (so the healthcare costs of a typical worker have the form $B \cdot (1.04)^t$ where $B$ is the cost today). Suppose that today's workers can afford their healthcare ($A$ is much bigger than $B$). Will that be always true? Why or why not?
**Solution:** Asymptotically $(1.04)^t$ will dominate $1.02^t$ for large $t$, so eventually our assumptions must break down.

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(e) Three strains of a contagion are spreading in a population, spreading at rates 1.05, 1.1, and 0.98 respectively. The total number of cases at time $t$ behaves like

$$A \cdot 1.05^t + B \cdot 1.1^t + C \cdot 0.98^t \,.$$

($A, B, C$ are constants). Which strain dominates eventually? What would the number of infected people look like?
**Solution:** When $t$ is large, $(0.98)^t$ is actually decaying so this strain will disappear. On the other hand since $1.1 > 1.05$ over time $1.1^t$ will be much bigger than

(f) Are the expressions $e^{2x+1}$ and $e^{2x}$ asymptotic as $x \to \pm\infty$? What about $e^{(x+1)^2}$ and $e^{x^2}$?
**Solution:** $e^{2x+1} = e \cdot e^{2x}$ so the the first is always larger. Similarly $e^{(x+1)^2} = e^{2x+1} \cdot e^{x^2}$ so the ratio of the two expressions tends to $\infty$ as $x \to \infty$ and to 0 as $x \to -\infty$.

(4) The interaction between two molecules is often modeled by the *Lennard-Jones potential*

$$V(r) = \epsilon \left[ \left(\frac{r}{R}\right)^{-12} - 2\left(\frac{r}{R}\right)^{-6} \right]$$

where $\epsilon, R > 0$ are constants and $r > 0$ is the distance between the molecules. Which term dominates as $r \to 0^+$? As $r \to \infty$? Sketch the potential.

**Solution:** As $r \to \infty$ the power law of index $-12$ decays faster than the power law of index 6, so $V(r) \sim -2\epsilon \left(\frac{r}{R}\right)^{-6}$ (and in particular is negative). When $r \to 0^+$ the power law of index $-12$ blows up faster than the power law of index $-6$, so $V(r) \sim \epsilon \left(\frac{r}{R}\right)^{-12}$ (so blowing up to $\infty$). The graph must therefore come down from $\infty$ near 0, dip below the axis, and then approach the axis from below as it decays toward zero when $r \to \infty$.

(5) The (attractive) interaction between two hadrons (say protons) due to the strong nuclear force can be modeled by the *Yukawa potential* $V_{\mathrm{Y}}(r) = -g^2 \frac{e^{-\alpha m r}}{r}$ where $r$ is the separation between the particles, and $g, \alpha, m$ are positive constants. The electrical repulsion between two protons is described by the Columb potential $V_{\mathrm{C}}(r) = kq^2 \frac{1}{r}$ where $k, q$ are also positive constants. Which interaction will dominate for large distances? Will the net interaction be attractive or repulsive? Note that $g^2$ is much larger than $kq^2$.

**Solution:** At large distances the exponentially decaying factor will suppress the strong interaction, making the electrical interaction dominate. This is why nuclear fusion requires such high temperatures: we need to get the protons **really close** to each other for the strong force to take over, and this requires them moving very fast or the electrical repulsion will keep them apart.

## 3. Asymptotics of complicated expressions

(6) Describe the following expressions in words

(a) $x + \log x$
**Solution:** This is the sum of $x$ and of the logarithm of $x$.

(b) $e^{|x-5|^3}$
**Solution:** This is the exponential, of the cube, of the absolute value, of $x - 5$.

(c) $\frac{1+x}{1+2x-x^2}$
**Solution:** This is the ratio of (the sum of 1 and $x$) and (the sum of 1, $2x$, and $-x^2$).

(d) $\frac{e^x + A\sin x}{e^x - x^2}$
**Solution:** This is the ratio of (the sum of $e^x$ and the product of $A$ and $\sin x$) and (the difference of $e^x$ and $x^2$).

(e) $\frac{Ae^{rt} + Be^{-st}}{t+t^2}$ where $r, s > 0$ and $A, B \neq 0$.
**Solution:** This is the sum of $A$ times the exponential of $r$ times $t$ and $B$ times the exponential of $-s$ times $t$, all divided by the sum of $t$ and $t^2$.

(7) For each of the functions above determine its asymptotics near 0 and near $+\infty$.

(a)
**Solution:** (a) As $x \to \infty$ the linear term dominates and $x + \log x \sim x$. As $x \to 0^+$, on the other hand, $x$ remains bounded while $\log x$ blows up to $-\infty$ so it dominates and $x + \log x \sim \log x$.

(b)

**Solution:** (b) For $x$ close to 0, $x-5 \sim -5$ so $|x-5| \sim 5$ so $|x-5|^3 \sim 125$ so $e^{|x-5|^3} \sim e^{125}$. For $x$ very large $x-5 \sim x$ and since $x$ is positive $|x-5| \sim |x| = x$ so $|x-5|^3 \sim x^3$. $e^{|x-5|^3}$ therefore grows roughly like $e^{x^3}$ (in truth $e^{x^3}$ is actually much bigger than $e^{(x-5)^3}$ – the ratio is on the scale of $e^{15x^2}$– but our expression captures the gist of the growth pattern).

(c)

**Solution:** (c) As $x \to 0$ $x, x^2$ are negligible next to the 1 so $\frac{1+x}{1+2x-x^2} \sim \frac{1}{1} = 1$. As $x \to \infty$ $x$ dominates 1 so $x+1 \sim x$ and $x^2$ dominates $x, 1$ so $1+2x-x^2 \sim -x^2$. Thus $\frac{1+x}{1+2x-x^2} \sim \frac{x}{-x^2} = -\frac{1}{x}$ – in other words the whole expression decays roughly like $\frac{1}{x}$.

(d)

**Solution:** (d) For $x$ near 0 we have $e^x \sim e^0 = 1$ and $\sin x \to 0$ (we'll later learn that $\sin x \sim x$ near 0) so $e^x + A\sin x \sim 1$ near 0. Similarly $x^2 \sim 0$ so $e^x - x^2 \sim 1$ and we have $\frac{e^x + A\sin x}{e^x - x^2} \sim \frac{1}{1} = 1$. For large $x$ we have $|\sin x| \leq 1$ so $A\sin x$ is much smaller than $e^x$ and $e^x + A\sin x \sim e^x$. Simliarly $e^x$ dominates any polynomial including $x^2$ and we have $e^x - x^2 \sim e^x$. Thus at infinity $\frac{e^x + A\sin x}{e^x - x^2} \sim \frac{e^x}{e^x} = 1$.

(e)

**Solution:** (e) As $t \to 0$ we have $t^2 \ll t$ so $t + t^2 \sim t$. $e^{rt} \sim e^0 \sim e^{-st}$ so

$$\frac{Ae^{rt} + Be^{-st}}{t + t^2} \sim \frac{A+B}{t} .$$

As $t \to \infty$, $t^2 \gg t$ while $e^{rt} \gg e^{-st}$ (growing exponential dominates the decaying one!). Thus

$$\frac{Ae^{rt} + Be^{-st}}{t + t^2} \sim \frac{Ae^{rt}}{t^2} .$$

Conversely as $t \to -\infty$ we have $e^{-st} \gg e^{rt}$ so

$$\frac{Ae^{rt} + Be^{-st}}{t + t^2} \sim \frac{Be^{-st}}{t^2} .$$