Math 100A – WORKSHEET 1
EXPRESSIONS AND ASYMPTOTICS

1. The ladder of functions

(1) Classify the following functions into *power laws / power functions* and *exponentials*: x^3, πx^{102}, e^{2x}, $c\sqrt{x}$, $-\frac{8}{x}$, 7^x, $8\cdot 2^x$, $-\frac{1}{\sqrt{3}}\cdot\frac{1}{2^x}$, $\frac{9}{x^{7/2}}$, x^e, π^x, $\frac{A}{x^b}$.

(2) Order the following functions from small to large *asymptotically as $x\to\infty$:*
   (a) 1, $\sqrt{x}$, $x^{-1/2}$,$x^{1/3}$, e^x, $x^{-1/3}$, 10^6x^{2024}, e^{-x}, e^{x^2}, $\frac{2024}{x^{100}}$, 5^x, x.

   (b) Extra: add in $\log x$, $e^{\sqrt{x}}$, $(\log x)^2$, $\log\log x$, $\frac{1}{\log x}$.

   (c) Repeat (a), this time as $x\to 0^+$.

2. Asymptotics: simple expressions

(3) How does the each expression behave when x is large? small? what is x is large but negative? Sketch a plot
   (a) $1-x^2+x^4$ (“Mexican hat potential”)

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(b) $ax^3 - bx^5$ $(a, b > 0)$

(c) $e^x - x^4$

(d) Wages in some country grow at 2% a year (so the wage of a typical worker has the form $A \cdot (1.02)^t$ where t is measured in years and A is the wage today). The cost of healthcare grows at 4% a year (so the healthcare costs of a typical worker have the form $B \cdot (1.04)^t$ where B is the cost today). Suppose that today's workers can afford their healthcare (A is much bigger than B). Will that be always true? Why or why not?

(e) Three strains of a contagion are spreading in a population, spreading at rates 1.05, 1.1, and 0.98 respectively. The total number of cases at time t behaves like

$$A \cdot 1.05^t + B \cdot 1.1^t + C \cdot 0.98^t \, .$$

(A, B, C are constants). Which strain dominates eventually? What would the number of infected people look like?

(f) Are the expressions e^{2x+1} and e^{2x} asymptotic as $x \to \pm\infty$? What about $e^{(x+1)^2}$ and e^{x^2}?

(4) The interaction between two molecules is often modeled by the *Lennard-Jones potential*

$$V(r) = \epsilon \left[\left(\frac{r}{R}\right)^{-12} - 2\left(\frac{r}{R}\right)^{-6} \right]$$

where $\epsilon, R > 0$ are constants and $r > 0$ is the distance between the molecules. Which term dominates as $r \to 0^+$? As $r \to \infty$? Sketch the potential.

(5) The (attractive) interaction between two hadrons (say protons) due to the strong nuclear force can be modeled by the *Yukawa potential* $V_{\mathrm{Y}}(r) = -g^2 \frac{e^{-\alpha m r}}{r}$ where r is the separation between the particles, and g, α, m are positive constants. The electrical repulsion between two protons is described by the Columb potential $V_{\mathrm{C}}(r) = kq^2 \frac{1}{r}$ where k, q are also positive constants. Which interaction will dominate for large distances? Will the net interaction be attractive or repulsive? Note that g^2 is much larger than kq^2.

3. Asymptotics of complicated expressions

(6) Describe the following expressions in words

(a) $x + \log x$

(b) $e^{|x-5|^3}$

(c) $\frac{1+x}{1+2x-x^2}$

(d) $\frac{e^x + A\sin x}{e^x - x^2}$

(e) $\frac{Ae^{rt} + Be^{-st}}{t+t^2}$ where $r, s > 0$ and $A, B \neq 0$.

(7) For each of the functions above determine its asymptotics near 0 and near $+\infty$.

(a)

(b)

(c)

(d)

(e)