Math 100A – SOLUTIONS TO WORKSHEET 2
LIMITS. ASYMPTOTES, AND CONTIJUITY

(1) Review of asymptotics: analyze the expression $\frac{e^x+A\sin x}{e^x-x^2}$ as $x\to\infty$, $x\to 0$, $x\to-\infty$.
Solution: This is a ratio. As $x\to\infty$ e^x grows rapidly while $A\sin x$ is bounded, so $e^x+A\sin x\sim e^x$, while in the denominator e^x dominates x^2 so $e^x-x^2\sim e^x$ and we get $\frac{e^x+A\sin x}{e^x-x^2}\sim 1$. As $x\to 0$ $e^x+A\sin x$ is close to $1+0=1$ and e^x-x^2 is close to $1-0=1$ so $\frac{e^x+A\sin x}{e^x-x^2}\sim 1$ in that regime. Finally as $x\to-\infty$ e^x decays rapidly, so $e^x-x^2\sim -x^2$ which is large. But $A\sin x$ oscillates so there is no clear asymptotic.

1. Limits

(2) Either evaluate the limit or explain why it does not exist. Sketching a graph might be helpful.
(a) $\lim_{x\to5}\left(x^3-x\right)$
Solution: When the function is defined by expression the limit can be obtained by plugging in. $\lim_{x\to5}\left(x^3-x\right)=125-5=120$.
(b) $\lim_{x\to1}f(x)$ where $f(x)=\begin{cases}\sqrt{x} & 0\le x<1\\ 3 & x=1\\ 2-x^2 & x>1\end{cases}$.
Solution: $\lim_{x\to1^+}f(x)=\lim_{x\to1^+}(2-x^2)=2-1^2=1$ and $\lim_{x\to1^-}f(x)=\lim_{x\to1^-}\sqrt{x}=\sqrt{1}=1$ so

$$\lim_{x\to1}f(x)=1\,.$$

(c) $\lim_{x\to1}f(x)$ where $f(x)=\begin{cases}\sqrt{x} & 0\le x<1\\ 1 & x=1\\ 4-x^2 & x>1\end{cases}$.
Solution: $\lim_{x\to1^+}f(x)=\lim_{x\to1^+}(4-x^2)=4-1^2=3$ and $\lim_{x\to1^-}f(x)=\lim_{x\to1^-}\sqrt{x}=\sqrt{1}=1$ so the limit does not exist (but the one-sided limits do).

(3) Let $f(x)=\frac{x-3}{x^2+x-12}$.
(a) (Final 2014) What is $\lim_{x\to3}f(x)$?
Solution: $f(x)=\frac{x-3}{(x-3)(x+4)}=\frac{1}{x+4}$ so $\lim_{x\to3}f(x)=\frac{1}{3+4}=\boxed{\frac{1}{7}}$.
(b) What about $\lim_{x\to-4}f(x)$?
Solution: The limit does not exist: if x is very close to -4 then $x+4$ is very small and $\frac{1}{x+4}$ is very large. That said, when $x>-4$ we have $\frac{1}{x+4}>0$ and when $x<-4$ we have $\frac{1}{x+4}<0$ so (in the extended sense)

$$\lim_{x\to-4^+}\frac{1}{x+4}=+\infty$$
$$\lim_{x\to-4^-}\frac{1}{x+4}=-\infty\,.$$

(4) Evaluate
(a) $\lim_{x\to\infty}\frac{e^x+A\sin x}{e^x-x^2}$
Solution: By problem 1 this is 1.
(b) $\lim_{x\to0}\frac{e^x+A\sin x}{e^x-x^2}$
Solution: By problem 1 this is 1 also.

Date: 11/9/2024, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

(c) $\lim_{x\to-\infty} \frac{e^x+A\sin x}{e^x-x^2}$

Solution: By problem 1 the numerator is bounded while the denominator grows like x^2, so the whole expression tends to 0.

(5) Evaluate

(a) $\lim_{x\to 2} \frac{x+1}{4x^2-1}$

Solution: The expression is well-behaved at $x = 2$ so $\lim_{x\to 2} \frac{x+1}{4x^2-1} = \frac{2+1}{4\cdot 2^2-1} = \frac{3}{15} = \frac{1}{5}$.

(b) (Final, 2014) $\lim_{x\to-3^+} \frac{x+2}{x+3}$.

Solution: As $x \to -3$ the numerator is close to -1 and while the denominator goes to 0 so the whole expression blows up: we have $\frac{x+2}{x+3} \sim \frac{-1}{x+3}$. Now when $x > -3$ we have $x + 3 > 0$ so the whole expression is negative and $\lim_{x\to-3^+} \frac{x+2}{x+3} = \lim_{x\to-3^+} -\frac{1}{x+3} = -\infty$.

(c) $\lim_{x\to 1} \frac{e^x(x-1)}{x^2+x-2}$

Solution: $\lim_{x\to 1} \frac{e^x(x-1)}{x^2+x-2} = \lim_{x\to 1} \frac{e^x(x-1)}{(x-1)(x+2)} = \lim_{x\to 1} \frac{e^x}{x+2} = \frac{e^1}{1+2} = \frac{e}{3}$.

(d) $\lim_{x\to-2^-} \frac{e^x(x-1)}{x^2+x-2}$

Solution: As $x \to -2$ we have $\frac{e^x(x-1)}{x^2+x-2} = \frac{e^x(x-1)}{(x-1)(x+2)} = \frac{e^x}{x+2} \sim \frac{e^{-2}}{x+2}$ and the expression blows up (we have a vertical asymptote). If $x < -2$ then $x + 2 < 0$ and thus

$$\lim_{x\to-2^-} \frac{e^x(x-1)}{x^2+x-2} = -\infty\,.$$

(e) $\lim_{x\to 1} \frac{1}{(x-1)^2}$

Solution: The function blows up at both sides, and remains positive on both sides. Therefore

$$\lim_{x\to 1} \frac{1}{(x-1)^2} = \infty\,.$$

(f) $\lim_{x\to 4} \frac{\sin x}{|x-4|}$

Solution: $|x-4| \to 0$ as $x \to 4$ while $\sin x \xrightarrow[x\to 4]{} \sin 4 \neq 0$, so the function blows up there. Since $|x-2|$ is positive and $\sin 4$ is negative ($\pi < 4 < 2\pi$) we have

$$\lim_{x\to 4} \frac{\sin x}{|x-4|} = -\infty\,.$$

(g) $\lim_{x\to\frac{\pi}{2}^+} \tan x$, $\lim_{x\to\frac{\pi}{2}^-} \tan x$.

Solution: We have $\tan x = \frac{\sin x}{\cos x}$. Now for x close to $\frac{\pi}{2}$, $\sin x$ is close to $\sin\frac{\pi}{2} = 1$, so $\sin x$ is positive. On the other hand $\lim_{x\to\frac{\pi}{2}} \cos x = \cos\frac{\pi}{2} = 0$ so $\tan x$ blows up there. Since $\cos x$ is decreasing on $[0, \pi]$ it is positive if $x < \frac{\pi}{2}$ and negative if $x > \frac{\pi}{2}$, so:

$$\lim_{x\to\frac{\pi}{2}^+} \tan x = -\infty$$
$$\lim_{x\to\frac{\pi}{2}^-} \tan x = +\infty$$

2. Asymptotes

(6) For each expression, determine its vertical and horizontal asymptotes.

(a) $\frac{x^2+1}{x-3}$

Solution: As $x \to \pm\infty$ we have $\frac{x^2+1}{x-3} \sim \frac{x^2}{x} \sim x$ so $\lim_{x\to\infty} \frac{x^2+1}{x-3} = \infty$ and there is no horizontal asymptote. As $x \to 3$ we have $x^2 + 1 \sim 10$ so $\frac{x^2+1}{x-3} \sim \frac{10}{x-3}$ which blows up at that point so we have a vertical asymptote.

(b) (Final, 2015) $\frac{x+1}{x^2+2x-8}$

Solution: As $x \to \pm\infty$ we have $\frac{x+1}{x^2+2x-8} \sim \frac{x}{x^2} \sim \frac{1}{x}$ so $\lim_{x\to-\infty} \frac{x+1}{x^2+2x-8} = 0$ and we have a horizontal asymptote $y = 0$ at both ends. Since $x^2 + 2x - 8 = (x-2)(x+4)$ we have $\frac{x+1}{x^2+2x-8} \sim \frac{3}{6(x-2)}$ as $x \to 2$ and $\frac{x+1}{x^2+2x-8} \sim \frac{-3}{-6(x+4)} = \frac{1}{2(x+4)}$ as $x \to -4$ we have and blowup and vertical asymptotes at $x = 2$ and $x = -4$.

(7) (Quiz, 2015) Evaluate $\lim_{x\to-\infty}\frac{3x}{\sqrt{4x^2+x}-2x}$

Solution: As $x\to-\infty$ since $\sqrt{x^2}=|x|=-x$ we have

$$\frac{3x}{\sqrt{4x^2+x}-2x}\sim\frac{3x}{\sqrt{4x^2}-2x}\sim\frac{3x}{2\,|x|-2x}$$
$$\sim\frac{3x}{2(-x)-2x}\sim\frac{3x}{-4x}=-\frac{3}{4}\,.$$

and hence $\lim_{x\to-\infty}\frac{3x}{\sqrt{4x^2+x}-2x}=\boxed{-\frac{3}{4}}$.

Solution: Change variables via $x=-y$ with $y\to\infty$. We are then looking at

$$\frac{-3y}{\sqrt{4y^2-y}+2y}\sim-\frac{3y}{\sqrt{4y^2}+2y}\sim-\frac{3y}{2y+2y}$$
$$\sim-\frac{3y}{4y}\sim-\frac{3}{4}\,.$$

and hence $\lim_{x\to-\infty}\frac{3x}{\sqrt{4x^2+x}-2x}=\boxed{-\frac{3}{4}}$.

3. Continuity

(8) Determine where each expression/function is continuous.

(a) $f(x)=\frac{x}{x}$. Can you "fix" the problem at 0?
Solution: This expression makes sense when $x\neq 0$, so it's continuous on the punctured axis $(-\infty,0)\cup(0,\infty)$. We also have $\lim_{x\to 0}\frac{x}{x}=1$ so defining $f(0)=1$ will "remove" the hole at zero and give a function continuous on the entire axis (in fact the function which is identically 1).

(b) ("Heaviside step function") $H(x)=\begin{cases}1 & x>0\\ 1/2 & x=0\\ 0 & x<0\end{cases}$. Can you "fix" the problem at 0?
Solution: On each ray $(-\infty,0),(0,\infty)$ the function is constant, hence continuous. Since $\lim_{x\to 0^-}H(x)=\lim_{x\to 0^-}0=0$ while $\lim_{x\to 0^+}H(x)=\lim_{x\to 0^+}1=1$ the one-sided limits exist but disagree (a "jump discontinuity") and no definition of $H(0)$ will make the function continuous. The function is thus continuous exactly on $(-\infty,0)\cup(0,\infty)$.

(c) $g(x)=\sqrt{\log x}$
Solution: The logarithm makes sense when $x>0$, but when $0<x<1$ we have $\log x<0$ so the square root won't make sense. The expression thus makes sense on $[1,\infty)$ and that's where the function will be continuous.

(9) ("Gluing functions") In each problem find the value of the constant k such that the function is continuous.

(a) $f(x)=\begin{cases}\frac{x^3-2x^2}{x-2} & x\neq 2\\ k & x=2\end{cases}$
Solution: For $x\neq 2$ we have $f(x)=\frac{x^2(x-2)}{x-2}=x^2$ so $\lim_{x\to 2}f(x)=\lim_{x\to 2}x^2=4$ and setting $\boxed{k=4}$ makes the function continuous at 2. It is continuous everywhere else since for any $x\neq 2$ f is defined by formula around x.

(b) $g(x)=\begin{cases}8-kx & x<k\\ x^2 & x\geq k\end{cases}$
Solution: The only relevant point is $x=k$. At that point the value of the expression on the left is $8-k^2$, on the right is k^2, so the function will be continuous when $8-k^2=k^2$, equivalently when $2k^2=8$ or $k^2=4$, that is when $\boxed{k=\pm 2}$.

(c) $h(x)=\begin{cases}Ax^2+Bx & x\leq k\\ Akx+D & x>k\end{cases}$ (here A,B,D are constants with $B\neq 0$)

Solution: The only relevant point is $x = k$. At that point the value of the expression on the left is $Ak^2 + Bk$, on the right is $Ak^2 + D$, so the function will be continuous when $Ak^2 + Bk = Ak^2 + D$, equivalently when $\boxed{k = \frac{D}{B}}$.

(d) $j(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x > 0 \\ k + \cos x & x \leq 0 \end{cases}$

Solution: Since the expression on the right doesn't make sense at $x = 0$ we can't just glue and need to resort to the definition. As $x \to 0$ we have that $\sin\left(\frac{1}{x}\right)$ is bounded while $x^2 \to 0$, so $x^2 \sin\left(\frac{1}{x}\right)$ will still decay to zero and $\lim_{x\to 0^+} j(x) = \lim_{x\to 0^+} x^2 \sin\left(\frac{1}{x}\right) = 0$. On the other side we have $k + \cos 0 = k + 1$ and the function will be continuous when this vanishes, in otherwords when $\boxed{k = -1}$.