Math 100A – SOLUTIONS TO WORKSHEET 5
THE CHAIN RULE

1. The Chain Rule

(1) We know $\frac{d}{dy}\sin y = \cos y$.
  (a) Expand $\sin(y+k)$ to first order in k. Write down the linear approximation to $\sin y$ about $y = a$.
  Solution: $\sin(y+k) \approx \sin y + k\cos y$ and $\sin y \approx \sin a + (y-a)\cos a$.
  (b) Now let $F(x) = \sin(3x)$. Expand $F(x+h)$ to linear order in h. What is the derivative of $\sin 3x$?
  Solution: $F(x+h) = \sin(3(x+h) = \sin(3x+3h)$ so we use $y = 3x$ in the previous example to get

$$\begin{aligned} F(x+h) &= \sin\left(3(x+h)\right) \\ &= \sin\left(3x+3h\right) \\ &\approx \sin(3x) + (3h)\cos(3x) \\ &= \sin(3x) + (3\cos(3x))h \end{aligned}$$

  so the derivative is $\boxed{3\cos(3x)}$.

(2) Write each function as a composition and differentiate
  (a) e^{3x}
  Solution: This is $f(g(x))$ where $g(x) = 3x$ and $f(y) = e^y$. The derivative is thus

$$e^{3x} \cdot \frac{\mathrm{d}(3x)}{\mathrm{d}x} = \boxed{3e^{3x}}.$$

  (b) $\sqrt{2x+1}$
  Solution: This is $f(g(x))$ where $g(x) = 2x+1$ and $f(y) = \sqrt{y}$. Thus

$$\frac{\mathrm{d}f(g(x))}{\mathrm{d}x} = f'(g(x))g'(x) = \frac{1}{2\sqrt{g}} \cdot 2 = \frac{1}{\sqrt{2x+1}}\,.$$

  (c) (Final, 2015) $\sin(x^2)$
  Solution: This is $f(g(x))$ where $g(x) = x^2$ and $f(y) = \sin y$. The derivative is then

$$\cos(x^2) \cdot 2x = 2x\cos(x^2)\,.$$

  (d) $(7x + \cos x)^n$.
  Solution: This is $f(g(x))$ where $g(x) = 7x + \cos x$ and $f(y) = y^n$. The derivative is thus

$$n\left(7x+\cos x\right)^{n-1} \cdot (7 - \sin x)\;.$$

(3) (Final, 2012) Let $f(x) = g(2\sin x)$ where $g'(\sqrt{2}) = \sqrt{2}$. Find $f'\left(\frac{\pi}{4}\right)$.
  Solution: By the chain rule, $f'(x) = g'(2\sin x) \cdot \frac{\mathrm{d}}{\mathrm{d}x}(2\sin x) = 2g'(2\sin x)\cos x$. In particular,

$$\begin{aligned} f'\left(\frac{\pi}{4}\right) &= 2g'\left(2\sin\frac{\pi}{4}\right)\cos\frac{\pi}{4} = 2g'\left(2\frac{\sqrt{2}}{2}\right)\cdot\frac{\sqrt{2}}{2} \\ &= 2\sqrt{2}\cdot\frac{\sqrt{2}}{2} = 2. \end{aligned}$$

(4) Differentiate

Date: 2/10/2024, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

(a) a^x for fixed $a > 0$ (hint: $a = e^{\log a}$)
Solution: We have $a^x = \left(e^{\log a}\right)^x = e^{(\log a)\cdot x}$so

$$\begin{aligned}\frac{d(a^x)}{dx} &= \frac{d\left(e^{x\log a}\right)}{d(x\log a)}\cdot\frac{d(x\log a)}{dx}\\ &= e^{x\log a}\cdot\log a = \boxed{e^x\log a}.\end{aligned}$$

(b) $7x + \cos(x^n)$
Solution: We apply linearity and then the chain rule:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(7x+\cos(x^n)\right) &= \frac{\mathrm{d}(7x)}{\mathrm{d}x}+\frac{\mathrm{d}\cos(x^n)}{\mathrm{d}x}\\ &= 7+\frac{\mathrm{d}\cos(x^n)}{\mathrm{d}(x^n)}\cdot\frac{\mathrm{d}(x^n)}{\mathrm{d}x}\\ &= 7-\sin(x^n)\cdot nx^{n-1}.\end{aligned}$$

(c) $e^{\sqrt{\cos x}}$
Solution: We repeatedly apply the chain rule:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}e^{\sqrt{\cos x}} &= e^{\sqrt{\cos x}}\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{\cos x}\\ &= e^{\sqrt{\cos x}}\frac{1}{2\sqrt{\cos x}}\frac{\mathrm{d}}{\mathrm{d}x}\cos x\\ &= -e^{\sqrt{\cos x}}\frac{\sin x}{2\sqrt{\cos x}}.\end{aligned}$$

(d) $\left(\tan(e^{-x^2})\right)^3$
Solution: We repeatedly apply the chain rule:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\tan(e^{-x^2})\right)^3 &= 3\left(\tan(e^{-x^2})\right)^2\frac{\mathrm{d}}{\mathrm{d}x}\tan(e^{-x^2})\\ &= 3\left(\tan(e^{-x^2})\right)^2\left(1+\tan^2(e^{-x^2})\right)\frac{\mathrm{d}}{\mathrm{d}x}e^{-x^2}\\ &= 3\left(\tan(e^{-x^2})\right)^2\left(1+\tan^2(e^{-x^2})\right)e^{-x^2}\frac{\mathrm{d}}{\mathrm{d}x}(-x^2)\\ &= \boxed{-6\left(\tan(e^{-x^2})\right)^2\left(1+\tan^2(e^{-x^2})\right)e^{-x^2}x}.\end{aligned}$$

(e) (Final 2012) $e^{(\sin x)^2}$
Solution: We repeatedly apply the chain rule:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{(\sin x)^2}\right) &= e^{(\sin x)^2}\frac{\mathrm{d}}{\mathrm{d}x}\left((\sin x)^2\right)\\ &= e^{(\sin x)^2}2\sin x\frac{\mathrm{d}}{\mathrm{d}x}\sin x\\ &= e^{(\sin x)^2}2\sin x\cos x\\ &= e^{(\sin x)^2}\sin(2x).\end{aligned}$$

(f) $\sqrt{x+\sqrt{x+\sqrt{x}}}$

Solution: By the chain rule:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{x+\sqrt{x+\sqrt{x}}}\right) &= \frac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}}\frac{\mathrm{d}}{\mathrm{d}x}\left(x+\sqrt{x+\sqrt{x}}\right) \\ &= \frac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}}\cdot\left[1+\frac{1}{2\sqrt{x+\sqrt{x}}}\frac{\mathrm{d}}{\mathrm{d}x}\left(x+\sqrt{x}\right)\right] \\ &= \frac{1}{2\sqrt{x+\sqrt{x+\sqrt{x}}}}\cdot\left[1+\frac{1}{2\sqrt{x+\sqrt{x}}}\left\{1+\frac{1}{2\sqrt{x}}\right\}\right].\end{aligned}$$

(5) Suppose f, g are differentiable functions with $f(g(x)) = x^3$. Suppose that $f'(g(4)) = 5$. Find $g'(4)$.
Solution: Applying the chain rule we have $f'(g(x))\cdot g'(x) = 3x^2$. Plugging in $x = 4$ we get $5g'(4) = 3\cdot 4^2$ and hence $g'(4) = \frac{48}{5}$.

2. Logarithmic differentiation

(6) $\log\left(e^{10}\right) =$ $\quad\quad$ $\log(2^{100}) =$
Solution: $\log e^{10} = 10$ while $\log(2^{100}) = 100\log 2$.

(7) Differentiate
(a) $\frac{\mathrm{d}(\log(ax))}{\mathrm{d}x} =$ $\quad\quad$ $\frac{\mathrm{d}}{\mathrm{d}t}\log\left(t^2+3t\right) =$
Solution: By the chain rule, the derivatives are: $\frac{1}{ax}\cdot a = \frac{1}{x}$ and $\frac{1}{t^2+3t}\cdot(2t+3) = \frac{2t+t}{t^2+3t}$. We can also use the logarithm laws first: $\log(ax) = \log a + \log x$ so $\frac{\mathrm{d}}{\mathrm{d}x}(\log ax) = \frac{\mathrm{d}}{\mathrm{d}x}(\log a) + \frac{\mathrm{d}}{\mathrm{d}x}(\log x) = \frac{1}{x}$ since $\log a$ is constant if a is. Similarly, $\log(t^2+3t) = \log t + \log(t+3)$ so its derivative is $\frac{1}{t}+\frac{1}{t+3}$.
(b) $\frac{\mathrm{d}}{\mathrm{d}x}x^2\log(1+x^2) =$ $\quad\quad$ $\frac{\mathrm{d}}{\mathrm{d}r}\frac{1}{\log(2+\sin r)} =$
Solution: Applying the product rule and then the chain rule we get: $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\log(1+x^2)\right) = 2x\log(1+x^2) + x^2\frac{1}{1+x^2}\cdot 2x = 2x\log(1+x^2) + \frac{2x^3}{1+x^2}$. Using the quotient rule and the chain rule we get

$$\frac{\mathrm{d}}{\mathrm{d}r}\frac{1}{\log(2+\sin r)} = -\frac{1}{\log^2(2+\sin r)}\cdot\frac{1}{2+\sin r}\cdot\cos r = -\frac{\cos r}{(2+\sin r)\log^2(2+\sin r)}.$$

(8) (Logarithmic differentiation) differentiate
$y = \left(x^2+1\right)\cdot\sin x\cdot\frac{1}{\sqrt{x^3+3}}\cdot e^{\cos x}$.
Solution: We have

$$\begin{aligned}\log y &= \log\left(x^2+1\right) + \log(\sin x) + \log\left(\frac{1}{\sqrt{x^3+3}}\right) + \log\left(e^{\cos x}\right) \\ &= \log\left(x^2+1\right) + \log\left(\sin x\right) - \frac{1}{2}\log\left(x^3+3\right) + \cos x\,.\end{aligned}$$

Differentiating with respect to x gives:

$$\frac{y'}{y} = \frac{2x}{x^2+1} + \frac{\cos x}{\sin x} - \frac{1}{2}\frac{3x^2}{x^3+3} - \sin x$$

and solving for y' finally gives

$$y' = \left(\frac{2x}{x^2+1} + \frac{\cos x}{\sin x} - \frac{3x}{2(x^3+3)} - \sin x\right)\cdot\left(x^2+1\right)\cdot\sin x\cdot\frac{1}{\sqrt{x^3+3}}\cdot e^{\cos x}.$$

(9) Differentiate using $\boxed{f' = f\times(\log f)'}$
(a) x^n
Solution: If $y = x^n$ then $\log y = n\log x$. Differentiating with respect to x gives $\frac{1}{y}y' = \frac{n}{x}$ so $y' = y\frac{n}{x} = nx^{n-1}$.
Solution: By the rule, $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^n\right) = x^n\frac{\mathrm{d}}{\mathrm{d}x}\left(\log(x^n)\right) = x^n\left(\frac{n}{x}\right) = nx^{n-1}$.

(b) x^x

Solution: If $y = x^x$ then $\log y = x \log x$. Differentiating with respect to x gives $\frac{1}{y}y' = \log x + x \cdot \frac{1}{x} = \log x + 1$ so $y' = y(\log x + 1) = x^x(\log x + 1)$.

Solution: By the rule, $\frac{\mathrm{d}}{\mathrm{d}x}(x^x) = x^x \frac{\mathrm{d}}{\mathrm{d}x}(\log(x^x)) = x^x(\log x + 1)$.

Solution: We have $x^x = \left(e^{\log x}\right)^x = e^{x \log x}$. Applying the chain rule we now get $(x^x)' = e^{x\log x}(\log x + 1) = x^x(\log x + 1)$.

(c) $(\log x)^{\cos x}$

Solution: By the logarithmic differentiation rule we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\log x)^{\cos x} &= (\log x)^{\cos x} \cdot \frac{\mathrm{d}}{\mathrm{d}x}(\cos x \log(\log x)) \\ &= -\sin x \log\log x\,(\log x)^{\cos x} + (\log x)^{\cos x}\cos x \frac{1}{\log x}\frac{1}{x} \\ &= -\sin x \log\log x\,(\log x)^{\cos x} + \cos x\,(\log x)^{\cos x - 1}\frac{1}{x}.\end{aligned}$$

(d) (Final, 2014) Let $y = x^{\log x}$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x only.

Solution: By the logarithmic differentiation rule we have

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= y\frac{\mathrm{d}\log y}{\mathrm{d}x} = x^{\log x}\frac{\mathrm{d}}{\mathrm{d}x}(\log x \cdot \log x) \\ &= x^{\log x}\left(2\log x \cdot \frac{1}{x}\right) = 2\log x \cdot x^{\log x - 1}.\end{aligned}$$

3. More problems

(10) Let $f(x) = g(x)^{h(x)}$. Find a formula for f' in terms of g' and h'.

Solution: By the logarithmic differentiation rule we have

$$\begin{aligned}f' &= f \cdot (h \log g)' \\ &= f\left(h' \log g + \frac{h}{g}g'\right) \\ &= h \cdot g^{h-1} \cdot g' + g^h \log g \cdot h'.\end{aligned}$$

Observe that this is the sum of what we'd get by applying the power law rule and the exponential rule.

(11) Let $f(\theta) = \sin^2\theta + \cos^2\theta$. Find $\frac{df}{d\theta}$ without using trigonometric identities. Evaluate $f(0)$ and conclude that $\sin^2\theta + \cos^2\theta = 1$ for all θ.

Solution: By the chain rule $\frac{d}{d\theta}(\sin\theta)^2 = 2\sin\theta\cos\theta$ and $\frac{d}{d\theta}(\cos\theta)^2 = 2\cos\theta(-\sin\theta)$ so

$$\frac{df}{d\theta} = 2\sin\theta\cos\theta - 2\sin\theta\cos\theta = 0,$$

It follows that f is constant; since $f(0) = (\sin 0)^2 + (\cos 0)^2 = 1$ we have $f(\theta) = 1$ for all θ, which is the claim.

(12) ("Inverse function rule") suppose $f(g(x)) = x$ for all x.

(a) Show that $f'(g(x)) = \frac{1}{g'(x)}$.

Solution: Applying the chain rule we have $f'(g(x)) \cdot g'(x) = 1$.

(b) Suppose $g(x) = e^x$, $f(y) = \log y$. Show that $f(g(x)) = x$ and conclude that $(\log y)' = \frac{1}{y}$.

Solution: $f(g(x)) = \log(e^x) = x$. We then have $f'(e^x) = \frac{1}{g'(x)} = \frac{1}{e^x}$ so $f'(y) = \frac{1}{y}$ for all $y > 0$.

(c) Suppose $g(\theta) = \sin\theta$, $f(x) = \arcsin x$ so that $f(g(\theta)) = \theta$. Show that $f'(x) = \frac{1}{\sqrt{1-x^2}}$.

Solution: We have $f'(\sin\theta) = \frac{1}{\cos\theta} = \frac{1}{\sqrt{1-\sin^2\theta}}$ so $f'(x) = \frac{1}{\sqrt{1-x^2}}$ for $-1 < x < 1$.

(13) (Final, 2015) Let $xy^2 + x^2y = 2$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ at the point $(1, 1)$.

Solution: Differentiating with respect to x we find $y^2 + 2xy\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy + x^2\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ along the curve. Setting $x = y = 1$ we find that, at the indicated point,

$$3 + 3\frac{\mathrm{d}y}{\mathrm{d}x}\Big|_{(1,1)} = 0$$

so

$$\frac{\mathrm{d}y}{\mathrm{d}x}\Big|_{(1,1)} = -1\,.$$