Math 100A – WORKSHEET 5
THE CHAIN RULE

1. The Chain Rule

(1) We know $\frac{d}{dy}\sin y = \cos y$.
   (a) Expand $\sin(y+h)$ to linear order in h. Write down the linear approximation to $\sin y$ about $y = a$.

   (b) Now let $F(x) = \sin(3x)$. Expand $F(x+h)$ to linear order in h. What is the derivative of $\sin 3x$?

Fact. $(f(g(x)))' = f'(g(x))\,g'(x)$ *or* $\frac{\mathrm{d}}{\mathrm{d}x}(f(g(x))) = \frac{df}{dg}\cdot\frac{dg}{dx}$.

(2) Write each function as a composition and differentiate
   (a) e^{3x}

   (b) $\sqrt{2x+1}$

   (c) (Final, 2015) $\sin(x^2)$

   (d) $(7x+\cos x)^n$.

(3) (Final, 2012) Let $f(x) = g(2\sin x)$ where $g'(\sqrt{2}) = \sqrt{2}$. Find $f'\left(\frac{\pi}{4}\right)$.

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(4) Differentiate
   (a) a^x for fixed $a > 0$ (hint: $a = e^{\log a}$)

   (b) $7x + \cos(x^n)$

   (c) $e^{\sqrt{\cos x}}$

   (d) $\left(\tan(e^{-x^2})\right)^3$

   (e) (Final 2012) $e^{(\sin x)^2}$

   (f) $\sqrt{x + \sqrt{x + \sqrt{x}}}$

(5) Suppose f, g are differentiable functions with $f(g(x)) = x^3$. Suppose that $f'(g(4)) = 5$. Find $g'(4)$.

2. Logarithmic differentiation

$\boxed{\log_b(b^x) = b^{\log_b x} = x}$ $\boxed{\log_b(xy) = \log_b x + \log_b y}$ $\boxed{\log_b(x^y) = y \log_b x}$ $\boxed{\log_b \frac{1}{x} = -\log_b x}$

Fact. $\frac{d}{dx} \log x = \frac{1}{x}$

(6) $\log\left(e^{10}\right) =$ $\log(2^{100}) =$ (in terms of $\log 2$)

(7) Differentiate
   (a) $\frac{\mathrm{d}(\log(ax))}{\mathrm{d}x} =$ $\frac{\mathrm{d}}{\mathrm{d}t} \log\left(t^2 + 3t\right) =$

   (b) $\frac{\mathrm{d}}{\mathrm{d}x} x^2 \log(1 + x^2) =$ $\frac{\mathrm{d}}{\mathrm{d}r} \frac{1}{\log(2 + \sin r)} =$

(8) (Logarithmic differentiation) Use $\log(fg) = \log f + \log g$ to differentiate $y = \left(x^2+1\right) \cdot \sin x \cdot \frac{1}{\sqrt{x^3+3}} \cdot e^{\cos x}$.

(9) Differentiate using $\boxed{f' = f \times (\log f)'}$

(a) x^n

(b) x^x

(c) $(\log x)^{\cos x}$

(d) (Final, 2014) Let $y = x^{\log x}$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x only.

3. More problems

(10) Let $f(x) = g(x)^{h(x)}$. Find a formula for f' in terms of g' and h'.

(11) Let $f(\theta) = \sin^2\theta + \cos^2\theta$. Find $\frac{df}{d\theta}$ without using trigonometric identities. Evaluate $f(0)$ and conclude that $\sin^2\theta + \cos^2\theta = 1$ for all θ.

(12) ("Inverse function rule") suppose $f(g(x)) = x$ for all x.
(a) Show that $f'(g(x)) = \frac{1}{g'(x)}$.

(b) Suppose $g(x) = e^x$, $f(y) = \log y$. Show that $f(g(x)) = x$ and conclude that $(\log y)' = \frac{1}{y}$.

(c) Suppose $g(\theta) = \sin\theta$, $f(x) = \arcsin x$ so that $f(g(\theta)) = \theta$. Show that $f'(x) = \frac{1}{\sqrt{1-x^2}}$.

(13) (Final, 2015) Let $xy^2 + x^2y = 2$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ at the point $(1, 1)$.