Math 100A – SOLUTIONS TO WORKSHEET 6
INVERSE TRIGONOMETRIC FUNCTIONS

1. Review

(1) Differentiate
  (a) $e^{\sqrt{\cos x}}$
  Solution: We repeatedly apply the chain rule:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}e^{\sqrt{\cos x}} &= e^{\sqrt{\cos x}}\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{\cos x}\\ &= e^{\sqrt{\cos x}}\frac{1}{2\sqrt{\cos x}}\frac{\mathrm{d}}{\mathrm{d}x}\cos x\\ &= -e^{\sqrt{\cos x}}\frac{\sin x}{2\sqrt{\cos x}}\,.\end{aligned}$$

(2) (Final, 2014) Let $y = x^{\log x}$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x only.
  Solution: By the logarithmic differentiation rule we have

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= y\frac{\mathrm{d}\log y}{\mathrm{d}x} = x^{\log x}\frac{\mathrm{d}}{\mathrm{d}x}\left(\log x\cdot\log x\right)\\ &= x^{\log x}\left(2\log x\cdot\frac{1}{x}\right) = 2\log x\cdot x^{\log x-1}\,.\end{aligned}$$

(3) Find the line tangent to the curve $y^2 = 4x^3 + 2x$ at the point $(2, 6)$.
  Solution: Differentiating *along the curve* with respect to x we find $2y\frac{\mathrm{d}y}{\mathrm{d}x} = 12x^2 + 2$, so that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{6x^2+1}{y}$. In particular at the point $(2, 6)$ the slope is $\frac{25}{6}$ and the line is

$$y = \frac{25}{6}(x-2) + 6\,.$$

(4) (Final, 2015) Let $xy^2 + x^2y = 2$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ at the point $(1, 1)$.
  Solution: Differentiating with respect to x we find $y^2 + 2xy\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy + x^2\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ along the curve. Setting $x = y = 1$ we find that, at the indicated point,

$$3 + 3\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(1,1)} = 0$$

so

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(1,1)} = -1\,.$$

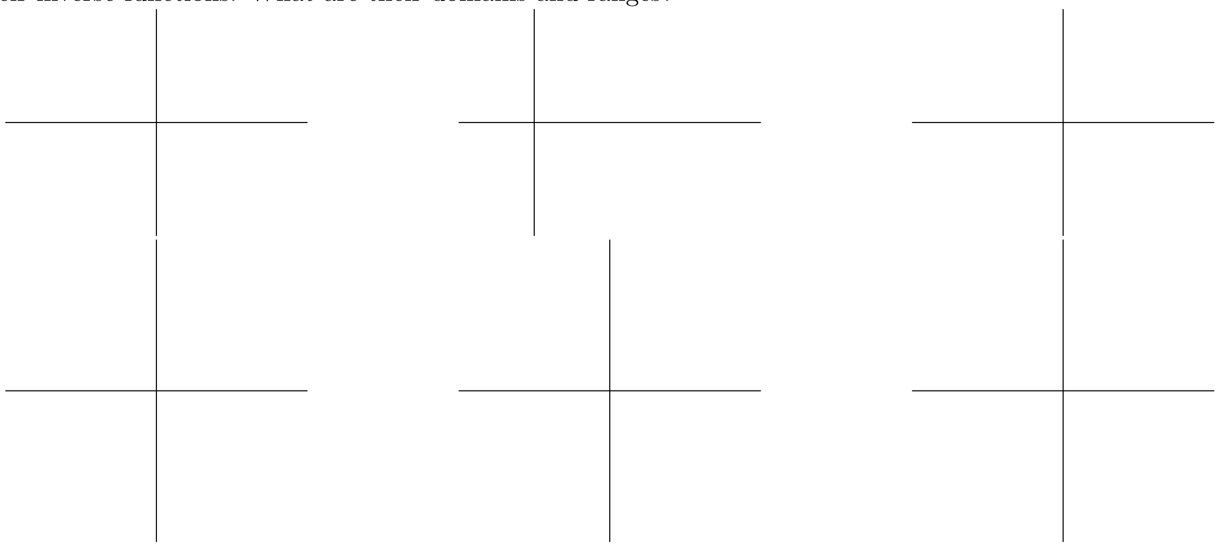
(5) (Final 2012) Find the slope of the line tangent to the curve $y + x\cos y = \cos x$ at the point $(0, 1)$.
  Solution: Differentiating with respect to x we find $y' + \cos y - x\sin y\cdot y' = -\sin x$, so that $y' = -\frac{\sin x+\cos y}{1-x\sin y} = \frac{\sin x+\cos y}{x\sin y-1}$. Setting $x = 0$, $y = 1$ we get that at that point $y' = \frac{\cos 1}{-1} = -\cos 1$.
(6) Find y'' (in terms of x, y) along the curve $x^5 + y^5 = 10$ (ignore points where $y = 0$).
  Solution: Differentiating with respect to x we find $5x^4 + 5y^4y' = 0$, so that $y' = -\frac{x^4}{y^4}$. Differentiating again we find

$$y'' = -\frac{4x^3}{y^4} + \frac{4x^4y'}{y^5} = -\frac{4x^3}{y^4} - \frac{4x^8}{y^9}\,.$$

Date: 9/10/2024, Worksheet by Lior Silberman.

2. Inverse trig functions

(7) Draw on the following axes graphs of $\sin\theta$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\cos\theta$ on $[0, \pi]$ and $\tan\theta$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, then of their inverse functions. What are their domains and ranges?



(8) Evaluation

(a) (Final 2014) Evaluate $\arcsin\left(-\frac{1}{2}\right)$ and $\arcsin\left(\sin\left(\frac{31\pi}{11}\right)\right)$.

Solution: $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$ so $\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$. Also $\sin\left(\frac{31\pi}{11}\right) = \sin\left(\frac{31\pi}{11} - 2\pi\right) = \sin\left(\frac{9\pi}{11}\right) = \sin\left(\pi - \frac{9\pi}{11}\right) = \sin\left(\frac{2\pi}{11}\right)$ and $\frac{2\pi}{11} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ so $\arcsin\left(\sin\left(\frac{31\pi}{11}\right)\right) = \frac{2\pi}{11}$.

(b) (Final 2015) Simplify $\sin(\arctan 4)$

Solution: Consider the right-angled triangle with sides 4, 1 and hypotenuse $\sqrt{1+4^2} = \sqrt{17}$. Let θ be the angle opposite the side of length 4. Then $\tan\theta = 4$ and $\sin\theta = \frac{4}{\sqrt{17}}$ so $\sin(\arctan 4) = \sin\theta = \frac{4}{\sqrt{17}}$.

(c) Find $\tan(\arccos(0.4))$

Solution: Consider the right-angled triangle with sides 0.4, $\sqrt{1-0.4^2}$ and hypotenuse 1. Let θ be the angle between the side of length 0.4 and the hypotenuse. Then $\cos\theta = \frac{0.4}{1} = 0.4$ and $\tan\theta = \frac{\sqrt{1-0.4^2}}{0.4} = \frac{\sqrt{0.84}}{0.4} = \sqrt{\frac{0.84}{0.16}} = \sqrt{5.25}$.

(9) Differentiation

(a) Find $\frac{\mathrm{d}}{\mathrm{d}x}(\arctan x)$

Solution: Let $\theta = \arctan x$. Then $x = \tan\theta$ so by the chain rule $1 = \frac{dx}{dx} = \frac{d\tan\theta}{dx} = \frac{d\tan\theta}{d\theta}\cdot\frac{d\theta}{dx} = (1+\tan^2\theta)\frac{d\theta}{dx}$ so

$$\frac{d(\arctan x)}{dx} = \frac{d\theta}{dx} = \frac{1}{1+\tan^2\theta} = \frac{1}{1+x^2}.$$

(b) Find $\frac{\mathrm{d}}{\mathrm{d}x}(\arcsin(2x))$

Solution: Since $\frac{\mathrm{d}}{\mathrm{d}x}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}$, the chain rule gives

$$\frac{\mathrm{d}}{\mathrm{d}x}(\arcsin(2x)) = \frac{2}{\sqrt{1-4x^2}}.$$

Alternatively, let $\theta = \arcsin 2x$, so that $\sin\theta = 2x$. Differentiating both sides we get

$$\cos\theta\cdot\frac{\mathrm{d}\theta}{\mathrm{d}x} = 2$$

so that

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{2}{\cos\theta} = \frac{2}{\sqrt{1-\sin^2\theta}} = \frac{2}{\sqrt{1-4x^2}}.$$

(c) Find the line tangent to $y = \sqrt{1 + (\arctan(x))^2}$ at the point where $x = 1$.

Solution: Since $\frac{\mathrm{d}}{\mathrm{d}x} \arctan(x) = \frac{1}{1+x^2}$, the chain rule gives

$$\frac{\mathrm{d}}{\mathrm{d}x} \sqrt{1 + (\arctan(x))^2} = \frac{1}{2\sqrt{1 + (\arctan(x))^2}} \cdot 2\arctan(x) \cdot \frac{1}{1+x^2}$$

$$= \frac{\arctan x}{(1+x^2)\sqrt{1 + (\arctan(x))^2}} .$$

Now $\arctan 1 = \frac{\pi}{4}$ so the line is

$$y = \frac{\pi}{8\sqrt{1 + \frac{\pi^2}{16}}} (x-1) + \sqrt{1 + \frac{\pi^2}{16}} .$$

(d) Find y' if $y = \arcsin\left(e^{5x}\right)$. What is the domain of the functions y, y'?

Solution: From the chain rule we get

$$\frac{\mathrm{d}}{\mathrm{d}x} \arcsin\left(e^{5x}\right) = \frac{1}{\sqrt{1 - e^{10x}}} 5e^{5x} = \frac{5e^{5x}}{\sqrt{1 - e^{10x}}} .$$

The function y itself is defined when $-1 \leq e^{5x} \leq 1$, that is when $5x \leq 0$, that is when $x \leq 0$. The derivative is defined when $e^{10x} < 1$, that is when $x < 0$. The point is that since $\sin\theta$ has horizontal tangents at $\pm\frac{\pi}{2}$, $\arcsin x$ has vertical tangents at ± 1.

Solution: We can write the identity as $\sin y = e^{5x}$ and differentiate both sides to get $y' \cos y = 5e^{5x}$ so that

$$y' = \frac{5e^{5x}}{\cos y} = \frac{5e^{5x}}{\sqrt{1 - \sin^2 y}} = \frac{5e^{5x}}{\sqrt{1 - e^{10x}}} .$$