Math 100A – SOLUTIONS TO WORKSHEET 7
APPLICATIONS OF THE CHAIN RULE

1. Related Rates 1: differentiation

(1) A particle is moving along the curve $y^2 = x^3 + 2x$. When it passes the point $\left(1, \sqrt{3}\right)$ we have $\frac{\mathrm{d}y}{\mathrm{d}t} = 1$. Find $\frac{\mathrm{d}x}{\mathrm{d}t}$.

Solution: We differentiate along the curve with respect to time, finding

$$2y\frac{\mathrm{d}y}{\mathrm{d}t} = 3x^2\frac{\mathrm{d}x}{\mathrm{d}t} + 2\frac{\mathrm{d}x}{\mathrm{d}t}\,.$$

Plugging in $\frac{\mathrm{d}y}{\mathrm{d}t} = 1$, $x = 1$, $y = \sqrt{3}$ we find: $2\sqrt{3} = 5\frac{\mathrm{d}x}{\mathrm{d}t}$ so at that time we have

$$\boxed{\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{2\sqrt{3}}{5}\,.}$$

(2) Air is pumped into a spherical balloon at the rate of $13\mathrm{cm}^3/\mathrm{s}$. How fast is the radius of the balloon changing when it is 15cm?

Solution: Let V be the volume, r the radius. We then have

$$V = \frac{4}{3}\pi r^3$$

and hence

$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}\,.$$

Plugging in $\frac{dV}{dt} = 13$, $r = 15$ we find

$$\boxed{\frac{dr}{dt} = \frac{13}{4\pi\cdot 15^2} = \frac{13}{900\pi}\frac{\mathrm{cm}}{\mathrm{s}}\,.}$$

(3) If we place an object at distance p from a thin lens, the *lenscrafter's formula* provides that the distance of the image from the lens, q, is determined by

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f},$$

where f is the *focal length* of the lens. Suppose that $f = 10\mathrm{cm}$ and $p = 30\mathrm{cm}$. If we move the object away from the lens at the rate of 4cm/s, how fast is the image moving? In which direction is it moving?

Solution: At the given instant we have $q = \frac{1}{\frac{1}{f} - \frac{1}{p}} = \frac{1}{\frac{1}{10} - \frac{1}{30}} = 15\mathrm{cm}$. Differentiating the relation we have (f is constant, being a property of the lens)

$$-\frac{\dot{p}}{p^2} - \frac{\dot{q}}{q^2} = 0$$

and hence

$$\dot{q} = -\frac{q^2}{p^2}\dot{p} = -\left(\frac{15}{30}\right)^2 \cdot 4 = -1\frac{\mathrm{cm}}{\mathrm{s}}\,.$$

The image is therefore moving toward the lens at $1\frac{\mathrm{cm}}{\mathrm{s}}$.

(4) The *ideal gas law* provides that $PV = NkT$ for mass of N particles of an ideal gas, where P is the pressure, V is the volume, T is the temperature, and k is Boltzmann's constant.

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(a) Suppose that we heat a mass of gas in a container of fixed volume. Relate the rate of change of the pressure to the rate of change of the temerature.
Solution: We have $V\frac{dP}{dt} = Nk\frac{dT}{dt}$ so $\frac{dP}{dt} = \frac{Nk}{V}\frac{dT}{dt}$.

(b) Suppose that we compress a piston while holding the temperature constant. Relate the rate of change of the pressure of the gas to the rate of change of its volume. Is the pressure increasing or decreasing?
Solution: We have $\frac{dP}{dt}V + P\frac{dV}{dt} = 0$ so $\frac{dP}{dt} = -\frac{P}{V}\frac{dV}{dt}$. We are given that $\frac{dV}{dt} < 0$ (the gas is being *compressed*), so $\frac{dP}{dt} > 0$ and the pressure is *increasing*.

(5) A closed rectangular box has sides of lengths $4, 5, 6$cm. Suppose that the first and second sides are lengthening by $2\frac{\text{cm}}{\text{sec}}$ while the third side is shortening by $3\frac{\text{cm}}{\text{sec}}$.

(a) How fast is the volume changing?
Solution: Call the sides x, y, z. The volume is then $V(x, y, z) = xyz$. By the product rule

$$\frac{dV}{dt} = \dot{x}yz + x\dot{y}z + xy\dot{z}$$

so at the given time

$$\begin{aligned}\dot{V} &= 2 \cdot 5 \cdot 6 + 4 \cdot 2 \cdot 6 + 4 \cdot 5 \cdot (-3) \\ &= \boxed{48\frac{\text{cm}^3}{\text{sec}}}\,.\end{aligned}$$

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(b) How fast is the surface area changing?
Solution: The surface area is $A(x, y, z) = 2xy + 2yz + 2zx$. By the product rule

$$\frac{dV}{dt} = 2\left[\dot{x}(y + x) + \dot{y}(x + z) + \dot{z}(x + y)\right]$$

so at the given time

$$\begin{aligned}\dot{A} &= 2\left[2(5 + 6) + 2(4 + 6) - 3(4 + 5)\right] \\ &= \boxed{30\frac{\text{cm}^2}{\text{sec}}}\,.\end{aligned}$$

(c) How fast is the main diagonal changing?
Solution: The main diagonal's length satisfies $L^2 = x^2 + y^2 + z^2$. Differentiating we get

$$2L\dot{L} = 2x\dot{x} + 2y\dot{y} + 2z\dot{z}\,.$$

At the given time $L = \sqrt{4^2 + 5^2 + 6^2} = \sqrt{77}$ and hence

$$\begin{aligned}\dot{L} &= \frac{1}{\sqrt{77}}\left[2 \cdot 4 + 2 \cdot 5 - 3 \cdot 6\right] \\ &= \boxed{0\frac{\text{cm}}{\text{sec}}}\,.\end{aligned}$$

2. Related Rates 2: problem-solving

(6) (Final, 2015, variant) A conical tank of water is 6m tall and has radius 1m at the top.

(a) The drain is clogged, and is filling up with rainwater at the rate of $5\text{m}^3/\text{min}$. How fast is the water rising when its height is 5m?
Solution: The water fills a conical volume inside the drain. Suppose that at time t the height of the water is $h(t)$ and the radius at the surface of the water is $r(t)$. Then by similar triangles

$$\frac{r(t)}{h(t)} = \frac{1}{6}\,.$$

We therefore have $r(t) = \frac{h(t)}{6}$. The volume of the water is therefore

$$V(t) = \frac{1}{3}\pi r^2 h = \frac{\pi}{108}h^3(t)\,.$$

Differentiating we find
$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\pi}{36} h^2(t) \frac{\mathrm{d}h}{\mathrm{d}t}\,.$$
In particular, if $\frac{\mathrm{d}V}{\mathrm{d}t} = 5\mathrm{m}^3/\mathrm{min}$ and $h = 5\mathrm{m}$ then
$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{36 \cdot 5}{\pi \cdot 5^2} = \frac{36}{5\pi} \frac{\mathrm{m}}{\mathrm{min}}\,.$$

(b) The drain is unclogged and water begins to drain at the rate of $(5 + \frac{\pi}{4})\mathrm{m}^3/\mathrm{min}$ (but rain is still falling). At what height is the water falling at the rate of $1\mathrm{m}/\mathrm{min}$?
Solution: We are now given $\frac{\mathrm{d}V}{\mathrm{d}t} = -\frac{\pi}{4} \frac{\mathrm{m}^3}{\mathrm{min}}$ and $\frac{\mathrm{d}h}{\mathrm{d}t} = -1 \frac{\mathrm{m}}{\mathrm{min}}$. Thus at the given time
$$h(t) = \sqrt{\frac{36 \frac{\mathrm{d}V}{\mathrm{d}t}}{\pi \frac{\mathrm{d}h}{\mathrm{d}t}}} = \sqrt{\frac{-36\pi}{4\pi(-1)}} = \sqrt{9} = 3\mathrm{m}\,.$$

(c) Repeat the problem with tank upside-down (vertex on top).
Solution: Again let h be the distance from the vertex to the surface of the water, and let $r(t)$ be the radius of the surface of the water. We again have $\frac{r(t)}{h(t)} = \frac{R}{H}$ so $r = \frac{R}{H}h$. However, the volume of the water is now $V(t) = \frac{1}{3}\pi R^2 H - \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi R^2 H - \frac{1}{3}\pi \frac{R^2}{H^2} h^3 = \frac{\pi R^2}{3H^2}\left(H^3 - h^3\right)$. Differentiating with respect to t then gives:
$$\frac{dV}{dt} = -\frac{\pi R^2}{H^2} h^2 \frac{dh}{dt}\,.$$
For (a) we have
$$\frac{d(H-h)}{dt} = -\frac{dh}{dt} = \frac{H^2}{\pi R^2 h^2} \frac{dV}{dt}\,,$$
and hence
$$\frac{d(H-h)}{dt} = \frac{6^2}{\pi \cdot 1^2 \cdot 1^2} \cdot 5 = \frac{36}{\pi} \frac{\mathrm{m}}{\mathrm{min}}\,.$$
For part (b) we are given $\frac{dV}{dt} = -\frac{\pi}{4}$, $\frac{d(H-h)}{dh} = -1$, so
$$-1 = \frac{6^2}{\pi \cdot 1^2 \cdot h^2} \cdot \left(-\frac{\pi}{4}\right)$$
so $h = 3\mathrm{m}$ and
$$H - h = 3\mathrm{m}\,.$$

(7) (Final, 2019) A 2m tall woman is running at night, moving away from a 6m-tall lamp-post. Her velocity t seconds after leaving the lamp-post is given (in metres per second) by $v(t) = 4 - \sin(2\pi t)$. How quickly is the length of her shadow changing after 3 seconds?

Solution: Say that at time t the woman's position is $x(t)$ metres away from the lamp-post, while her shadow is $s(t)$ metres long. Say her height is h and the height of the lamp-post is H. Let α be the angle by which the tip of the shadow sees the head of the woman and the lamp-post. Then
$$\tan\alpha = \frac{h}{s(t)} = \frac{H}{s(t) + x(t)}\,.$$
It follows that $\frac{H}{h}s(t) = s(t) + x(t)$ and hence $s(t) = \frac{1}{\frac{H}{h} - 1}x(t) = \frac{h}{H-h}x(t)$ .Differentiating with respect to time we get
$$\begin{aligned}\frac{ds}{dt} &= \frac{h}{H-h}\frac{dx}{dt} = \frac{h}{H-h}v(t) \\ &= \frac{h}{H-h}\left(4 - \sin(2\pi t)\right)\,.\end{aligned}$$
Plugging in $h = 2$, $H = 6$, $t = 3$ and noting that $\sin(6\pi) = 0$ we get
$$\boxed{\frac{ds}{dt}(3\mathrm{s}) = 2\frac{\mathrm{m}}{\mathrm{s}}}\,.$$

(8) (Final 2018)

(a) Particle A travels with a constant speed of 2 units per minute on the x-axis starting at the point $(4, 0)$ and moving away from the origin, while particle B travels with a constant speed of 1 unit per minute on the y-axis starting at the point $(0, 8)$ and moving towards the origin. Find the rate of change of the distance between the two particles when the distance between the two particles is exactly 10 units.

Solution: At time t the first particle is at $(4 + 2t, 0)$ and the second is at $(0, 8 - t)$. The distance between them therefore satisfies

$$\begin{aligned} d(t)^2 &= (4 + 2t)^2 + (8 - t)^2 \\ &= 16 + 16t + 4t^2 + 64 - 16t + t^2 \\ &= 80 + 5t^2 \,. \end{aligned}$$

First, $d(t)^2 = 100$ when $t = 2$. Second at that time

$$2d\dot{d} = 10t = 20$$

so $\boxed{\dot{d}(2) = 1}$.

(b) Same question, but swap the velocities of the particles (particle A moves along the y axis, particle B moves along the x-axis).

Solution: At time t the first particle is at $(4, -t)$ and the second is at $(2t, 8)$ so the distance is now

$$\begin{aligned} d(t)^2 &= (2t - 4)^2 + (8 + t)^2 \\ &= 80 + 5t^2 \end{aligned}$$

and from this point the question is the same.