Math 100A – SOLUTIONS TO WORKSHEET 8
CURVE SKETCHING

1. Convexity and Concavity

(1) Consider the curve $y = x^3 - x$.
  (a) Find the line tangent to the curve at $x = 1$.
  Solution: $\frac{dy}{dx} = 3x^2 - 1$ so the derivative at $x = 1$ is 2. Since $y(1) = 0$ the line is $Y = 2(X - 1)$.
  (b) Near $x = 1$, is the line above or below the curve? Hint: how does the slope of the curve behave to the right and left of the point?
  Solution: Since $x > 0$ the slope is increasing near $x = 1$, so to the right the function grows faster than the line, to the right is decreases slower than the line, and the line is below.
  Solution: We have $x^3 - x - 2(x-1) = (x-1)(x^2+x-2) = (x-1)^2(x+2) = 3(x-1)^2 + (x-1)^3$ which is positive for x close enough to 1. In the Taylor expansion lecture we'll talk more about the representation $x^3 - x = 2(x-1) + 3(x-1)^2 + (x-1)^3$.

(2) For each curve find its domain; where is it concave up or down? Where are the inflection points.
  (a) $y = x \log x - \frac{1}{2}x^2$.
  Solution: This is defined on $(0, \infty)$. We have $y' = \log x - 1 - x$ so $y'' = \frac{1}{x} - 1$. Thus $y'' > 0$ if $x < 1$, $y'' < 0$ if $x > 1$, and the function is concave up on $(0, 1)$, concave down on $(1, \infty)$ and has an inflection point at $x = 1$.
  (b) $y = \sqrt[3]{x}$.
  Solution: This is an odd root, which is defined (and continuous) on the entire line. We have $y' = \frac{1}{3}x^{-2/3}$ which is defined for $x \neq 0$ (the tangent line at $x = 0$ is vertical, as we can see by switching to the representation $x = y^3$). We then have $y'' = -\frac{2}{9}x^{-5/3}$ which is positive when $x < 0$ and positive when $x > 0$, so the function changes convacity at $x = 0$ and that is an inflection point.

2. Curve sketching

(3) Let $f(x) = \frac{x^2}{x^2+1}$ for which $f'(x) = \frac{2x}{(x^2+1)^2}$ and $f''(x) = \frac{2(1-3x^2)}{(x^2+1)^3}$.
  (a) What are the domain and intercepts of f? What are the asymptotics at $\pm\infty$? Are there any vertical asymptotes? What are the asymptotics there?
  Solution: The function is defined for all x (always have $x^2 + 1 > 0$). We have $f(0) = 0$ and conversely if $f(x) = 0$ then $x^2 = 0$ so $x = 0$. As $x \to \pm\infty$ we have

$$\frac{x^2}{x^2+1} \sim \frac{x^2}{x^2} = 1$$

  so we have the horizontal asymptote $y = 1$ in both ends.
  (b) What are the intervals of increase/decrease? The local and global extrema?
  Solution: Since $\frac{2}{(x^2+1)^2}$ is always positive, $f'(x) > 0$ when $x > 0$ and $f'(x) < 0$ when $x < 0$. Thus f is decreasing on $(-\infty, 0)$, increasing on $(0, \infty)$ and has a local (and global) minimum at $x = 0$.
  (c) What are the intervals of concavity? Any inflection points?
  Solution: Since $\frac{2}{(x^2+1)^3}$ is always positive, the sign of $f''(x)$ is the same as that of $1 - 3x^2$. In particular $f''(x) > 0$ when $1 - 3x^2 > 0$, that is when $3x^2 < 1$ so when $|x| < \frac{1}{\sqrt{3}}$. Conversely $f''(x) < 0$ when $1 - 3x^2 < 0$ that is when $|x| > \frac{1}{\sqrt{3}}$ or when $x \in \left(-\infty, -\frac{1}{\sqrt{3}}\right) \cup \left(\frac{1}{\sqrt{3}}, \infty\right)$. We thus have inflection points at $\pm\frac{1}{\sqrt{3}}$.

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(d) Sketch a graph of $f(x)$.

(4) Let $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ for which $f'(x) = -\frac{1}{\sqrt{2\pi\sigma^6}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} (x-\mu)$ and $f''(x) = \frac{1}{\sqrt{2\pi\sigma^6}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \left(\frac{(x-\mu)^2}{\sigma^2} - 1\right)$.

(a) What are the domain and intercepts of f? What are the asymptotics at $\pm\infty$? Are there any vertical asymptotes? What are the asymptotics there?

Solution: The function is defined for all x and is always positive. We have $f(0) = \frac{1}{\sqrt{2\pi\sigma^2}}$. For large x the function will decay rapidly (morally like $e^{-x^2/2\sigma^2}$ even if that's not the correct asymptotics), so we have the horizontal asymptote $y = 0$ on both sides.

(b) What are the intervals of increase/decrease? The local and global extrema?

Solution: Since $\frac{1}{\sqrt{2\pi\sigma^6}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ is always positive, $f'(x) > 0$ when $x < \mu$ and $f'(x) < 0$ when $x > \mu$. Thus f is increasing on $(-\infty, \mu)$, decreasing on (μ, ∞) and has a local (and global) maximum at $x = \mu$.

(c) What are the intervals of concavity? Any inflection points?

Solution: Since $\frac{1}{\sqrt{2\pi\sigma^{10}}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ is always positive, the sign of $f''(x)$ is the same as that of $\left((x-\mu)^2 - \sigma^2\right)$. In particular $f''(x) > 0$ when $|x - \mu| > \sigma$, that is on on $(-\infty, \mu - \sigma) \cup (\mu + \sigma, \infty)$. Conversely $f''(x) < 0$ when $|x - \mu| < \sigma$, that is on $(\mu - \sigma, \mu + \sigma)$. Finally we see there are inflection points at $\mu \pm \sigma$.

(d) Sketch a graph of $f(x)$.

(5) (Final, December 2007) Let $f(x) = x\sqrt{3-x}$.

(a) Find its domain, intercepts, and asymptotics at the endpoints.

Solution: The function is defined for if $3 - x \geq 0$ that is for $x \leq 3$. It is positive if $x > 0$, so if $0 < x < 3$ and negative if $x < 0$, and thus crosses the axis at $x = 0$. As $x \to -\infty$ we have $x\sqrt{3-x} \sim x\sqrt{-x} \sim -|x|^{3/2}$. As $x \to 3$ we have $x \sim 3(3-x)^{1/2}$.

(b) What are the intervals of increase/decrease? The local and global extrema?

Solution: We have $f'(x) = \sqrt{3-x} - \frac{x}{2\sqrt{3-x}} = \frac{2(3-x)-x}{2\sqrt{3-x}} = \frac{6-3x}{2\sqrt{3-x}} = \frac{3}{2} \cdot \frac{2-x}{\sqrt{3-x}}$. Since $\frac{3}{2\sqrt{3-x}}$ is always positive, the sign of $f'(x)$ is determined by $2 - x$. Thus f' is increasing on $x < 2$, decreasing for $2 < x < 3$ and has its unique local maximum at $x = 2$.

(c) Given $f''(x) = \frac{3x-12}{4}(3-x)^{-3/2}$, what are the intervals of concavity? Any inflection points?

Solution: We have $f''(x) = \frac{3(x-4)}{4(3-x)^{-3/2}}$ is always positive. Now the domain of the function is $x < 3$ so $x - 4 < -1 < 0$ on the entire domain and $f''(x) < 0$ for all x – so the function is concave down and has no inflection points.

(d) Sketch a graph of $f(x)$.