Math 100A – SOLUTIONS TO WORKSHEET 9
OPTIMIZATION

1. Optimization of functions

(1) Let $f(x) = x^4 - 4x^2 + 4$.

(a) Find the absolute minimum and maximum of f on the interval $[-5, 5]$.

Solution: $f'(x) = 4x^3 - 8x$ is defined everywhere and vanishes at $0, \pm\sqrt{2}$. We have $f(\pm 5) = 625 - 100 + 4 = 529$, $f(\pm\sqrt{2}) = 4 - 8 + 4 = 0$ and $f(0) = 4$. The global maximum is therefore 529, attained at ± 5 and the global minimum is 0, attained at $\pm\sqrt{2}$.

Takeaway: straightforward application of the calculus: the global maximum/minimum must either be at the end of the interval, or at the interior, and if in the interior must be at a critical or singular point.

(b) Find the absolute minimum and maximum of f on the interval $[-1, 1]$.

Solution: Now the only critical point is 0 (note that $\sqrt{2} > 1$). We have $f(\pm 1) = 1 - 4 + 4 = 1$ and $f(0) = 4$. The global maximum is now 4, attained at 0, while the global minimum is 1, attained at ± 1.

Takeaway: The interval matters. "Critical points", "singular points" etc mean *points in the interval*.

(c) Find the absolute minimum and maximum of f (if they exist) on the interval $(-1, 1)$.

Solution: The function still has a *local* maximum at 0 where $f(0) = 4$ while $f'(x) < 0$ on $(0, 1)$ and $f'(x) > 0$ on $(-1, 0)$ so this is also the global maximum. There is no global minimum since $f(x) > 1 = f(\pm 1)$ for any x in the open interval (x can get *close* to ± 1 but it can't actually *equal* them).

Takeaway: If the interval is open (does not include the endpoints) then there need not be a global maximum/minimum and we need to analyze the function more carefully – usually by finding the intervals where it's increasing/decreasing.

(d) Find the absolute minimum and maximum of f (if they exist) on the real line.

Solution: Since as $x \to \pm\infty$ we have $f(x) \sim x^4 \to \infty$ there is no global maximum. It also follows that outside some closed interval $f(x)$ only takes large values, so to find the minimum it's enough to consider a big closed interval $[-L, L]$. The minimum won't be at the endpoints (the function is large there) but rather at a critical point. By part (a) we see that the global minimum is 0 attained at $\pm\sqrt{2}$.

Takeaway: On an open interval a contintinuous function can tend to infinity. We can use that to not have to analyze it far enough away.

Solution: (Alternative) Since as $x \to \pm\infty$ we have $f(x) \sim x^4 \to \infty$ there is no global maximum. Next, For $x \geq 5$ we have $f'(x) = x(4x^2 - 8) \geq x(20 - 8) \geq 12x > 0$ and for $x \leq -5$ similarly $4x^2 - 8 \geq 12$ so $f(x) = x(4x^2 - 8) \leq 12x < 0$ (if $x \leq -5$ it's negative). Thus f is increasing if $x \geq 5$ and decreasing if $x \leq -5$. Thus it only take values smaller than $f(\pm 5) = 629$ inside the interval $[-5, 5]$ and the global minimum must be there.

Takeaway: We can make an explicit estimate on when f starts to be large (we use $|x| \geq 5$ here because in part (a) we analyzed what happens in $[-5, 5]$).

(2) Let $f(x) = |x|$. Find the absolute minimum and maximum of f on the interval $[-1, 3]$.

Solution: We have $f'(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$ so there are no critical points but there is a singular point at $x = 0$. We have $f(-1) = 1$, $f(0) = 0$, $f(3) = 3$ so the maximum of f on the interval is 3,

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attained at $x = 3$, and the minimum is 0, attained at $x = 0$.

Takeaway: Singular points matter too.

(3) Find the global extrema (if any) of $f(x) = \frac{1}{x}$ on the intervals $(0, 5)$ and $[1, 4]$.

Solution: Since f is defined and strictly decreasing on these intervals, there are no extrema in the first case (and in fact $\lim_{x\to 0^+} \frac{1}{x} = \infty$) but in the second case for any $1 \leq x \leq 4$ we have $1 = f(1) \geq f(x) \geq f(4) = \frac{1}{4}$ so 1 is the global maximum and $\frac{1}{4}$ is the global minimum.

Takeaway: can sometimes solve problems without calculus at all.

2. Optimization problems

(4) A fish swimming at speed v relative to the water faces a drag force of the form av^2 and thus has to output a power of av^3. If the fish is swimming against a current of speed $u > 0$ (thus with speed $v > u$), it will cover a distance L at time $\frac{L}{v-u}$. The total energy cost is then $E = av^3 \frac{L}{v-u}$. At what speed v should the fish swim to minimize this cost?

Solution: The function $E(v)$ is continuous and differentiable on its domain (u, ∞). As $v \to u^+$ the energy blows up (because the swimming time tends to infinity). As $v \to \infty$ we have $E(v) \sim aLv^2 \to \infty$. Since it takes large values towards the endpoints, we can look for the smallest value in a closed interval where a minimum must exist, necessarily at a critical point. We have

$$\begin{aligned} E'(v) &= \frac{aL}{(v-u)^2}\left[3v^2(v-u) - v^3\right] \\ &= \frac{2aLv^2}{(v-u)^2}\left[v - \frac{3}{2}u\right]. \end{aligned}$$

The only critical point is then at $v_0 = \frac{3}{2}u$ (this is indeed greater than u, so in the domain), so that is the optimal speed.

Takeaway: We can solve problems involving parameters. Also, when the domain is open we still have to deal with the endpoints; often *asymptotics* help with that.

(5) A standard model for the interaction between two neutral molecules is the *Lennard-Jones Potential* $V(r) = \epsilon\left[\left(\frac{r}{R}\right)^{-12} - 2\left(\frac{r}{R}\right)^{-6}\right]$. Here r is the distance between the molecules and $R, \epsilon > 0$ are parameters.

(a) What is the range of r values that makes sense?

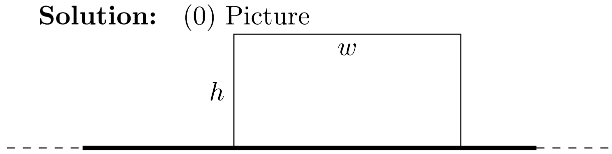
Solution: Distances are non-negative numbers. Since the potential blows up at $r = 0$ the we have $r \in (0, \infty)$.

(b) Physical systems tend to settle into a state of least energy. Find the minimum of this potential.

Solution: $V'(r) = \epsilon\left[-12R^{12}r^{-13} + 12R^6r^{-7}\right] = 12\epsilon\frac{R^{12}}{r^{13}}\left[\left(\frac{r}{R}\right)^6 - 1\right]$. We then see the potential decreasing for $\frac{r}{R} \in (0, 1)$ and increasing for $\frac{r}{R} \in (1, \infty)$, so the unique minimum is at $r = R$ where $V(R) = -\epsilon$.

(6) Suppose we have 100m of fencing to enlose a rectangular area against a long, straight wall. What is the largest area we can enclose?

Solution: (0) Picture



(1) Let the width of the rectangle be w, its height h, measured in metres.
(2) The total fencing used is then $2h + w$ so we must have $2h + w = 100$.
(3) The area of the rectangle is then $A = wh = h(100 - 2h)$.
(4) We must have $h \geq 0$ and since we have at most 100m of fencing we must have $h \leq 50$, so we need to optimize $A(h) = h(100 - 2h) = 100h - 2h^2$ on $[0, 50]$. We have $A'(h) = 100 - 4h$ which vanishes at $h = 25$. Since $A(0) = A(50) = 0$ (these are *degenerate rectangles*) and $A(25) = 25 \cdot 50 > 0$ $h = 25$m gives the maximum area.
(5) The maximum area we can enclose is 1250m^2.

(7) A ferry operator is trying to optimize profits. Before each ferry trip workers spend some time loading cars after which the trip takes 1 hour. The ferry can carry up to 100 cars, each paying \$50 for the trip. Worker salaries total \$500/hour and the fuel for the trip costs \$250. The workers can load $N(t) = 100\frac{t}{t+1}$ cars in t hours.

(a) How much time should be devoted to loading to maximize profits *per trip*.

Solution: If we load cars for t hours, we have revenues (in dollars) of $R(t) = 50N(t) = 5,000\frac{t}{t+1}$ and costs $C(t) = 250 + 500(1 + t) = 750 + 500t$ (the workers are paid for both loading the cars and for the trip; note the combination of fixed and variable costs). The profits are then

$$P(t) = R(t) - C(t) = 5000\frac{t}{t+1} - 500t - 750\,.$$

We note that $P(0) = -750$ (we lose money if we load no cars) and as $t \to \infty$ we have $P(t) \sim -500t \to -\infty$ (revenue is capped at 5000 – the loading time shows *diminishing returns*). Since $P(1) = 2500 - 500 - 750 > 0$ the maximum must be positive so somewhere in between, thus at a critical or singular point. We have $P(t) = 5000\frac{t+1}{t+1} - \frac{5000}{t+1} - 500t - 750 = 4250 - 5000\frac{1}{t+1} - 500t$ so $P'(t) = 5000\frac{1}{(1+t)^2} - 500$; this vanishes when $(1+t)^2 = 10$ so when $t_0 = \sqrt{10} - 1 \approx 2.16$ hours. We can also check that $P'(t) > 0$ if $t < t_0$ and $P'(t) < 0$ if $t > t_0$ so this really is a maximum.
Takeaway: If we are asked for the *time* there is no need to compute the precise profit at that time.

(b) The ferry runs continuously. How much time should be devoted to loading to maximize profits *per hour*?
Solution: If we load cars for t hours, our profits per hour are

$$\begin{aligned} Q(t) = \frac{P(t)}{t+1} &= 5000\frac{t}{(t+1)^2} - 500\frac{t}{t+1} - \frac{750}{t+1}\,. \\ &= \frac{5000t}{(t+1)^2} - \frac{250}{t+1} - 500 \\ &= \frac{4750t - 250}{(t+1)^2} - 500 \\ &= 250\frac{19t - 1}{(t+1)^2} - 500 \end{aligned}$$

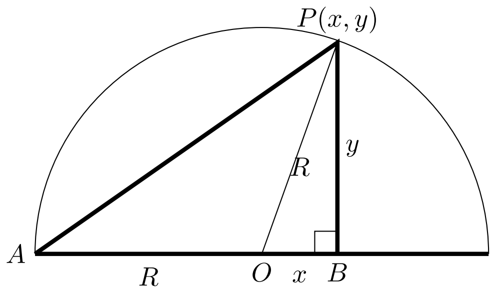
We note that $Q(0) = -750$ (we still lose money if we load no cars) and as $t \to \infty$ we have $Q(t) \sim \frac{-500t^2}{t^2} \to -500$ (if we load forever we just lose \$500 per hour paying the workers and make nothing on the trips). The maximum must therefore be somewhere in between, so at a critical or singular point. We have

$$\begin{aligned} Q'(t) &= 250\frac{19(t+1)^2 - (19t-1)2(t+1)}{(t+1)^4} \\ &= 250\frac{19(t+1) - (38t-2)}{(t+1)^3} = 250\frac{21 - 19t}{(t+1)^3}\,. \end{aligned}$$

This vanishes when $21 - 19t = 0$ so at $t_0 = \frac{21}{19} \approx 1.11$ hours; again we can verify that this is a maximum (e.g. by checking that $Q'(0) = 250 \cdot 9 > 0$ or by computing $Q(1) = 250 \cdot \frac{18}{4} - 500 = 250 \cdot 2\frac{1}{2} > 0$).
Takeaway: Changing the goal of the optimization can change the point of maximum: the time at which we maximize profits per hour is not the same as the time we maximize profits per trip.

(8) (Final 2012) The right-angled triangle ΔABP has the vertex $A = (-1, 0)$, a vertex P on the semicircle $y = \sqrt{1 - x^2}$, and another vertex B on the x-axis with the right angle at B. What is the largest possible area of such a triangle?

Solution: (0) Picture



(1) Put the coordinate system where the centre of the circle is at $(0,0)$ and the diameter is on the x-axis. Let B be at $(x,0)$, P at (x,y).
(2) Since P is on the circle we have $y=\sqrt{1-x^2}$. The area of the triangle is then $A=\frac{1}{2}(\text{base})\times(\text{height})=\frac{1}{2}(1+x)\sqrt{1-x^2}$ since the base of the triangle has length $1+x$.
(4) The function $A(x)$ is continuous on $[-1,1]$ so we can find its minimum by differentiation. By the product rule and chain rule,

$$\begin{aligned} A'(x) &= \frac{1}{2}\sqrt{1-x^2}+\frac{1}{2}(1+x)\frac{-2x}{2\sqrt{1-x^2}} \\ &= \frac{\left(\sqrt{1-x^2}\right)^2}{2\sqrt{1-x^2}}-\frac{x(1+x)}{2\sqrt{1-x^2}}=\frac{1-x^2-x-x^2}{2\sqrt{1-x^2}} \\ &= \frac{1-x-2x^2}{2\sqrt{1-x^2}}. \end{aligned}$$

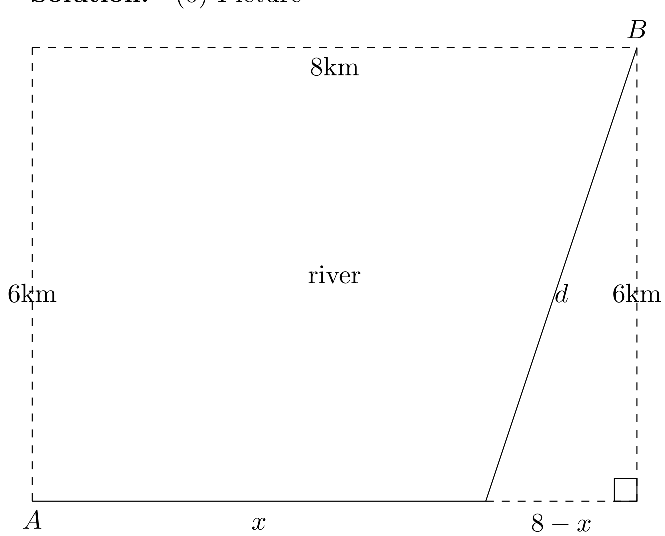
This is defined on $(-1,1)$ and the critical points satisfy $2x^2+x-1=0$ so they are $x=\frac{-1\pm\sqrt{1+8}}{4}=\frac{-1\pm 3}{4}=-1,\frac{1}{2}$. The only critical point in the interior is then $x=\frac{1}{2}$. The area vanishes at the endpoints (the triangle becomes degenerate) and

$$A\left(\frac{1}{2}\right)=\frac{1}{2}\cdot\frac{3}{2}\cdot\sqrt{1-\frac{1}{2^2}}=\frac{3\sqrt{3}}{8}.$$

It follows that the largest possible area is $\frac{3\sqrt{3}}{8}$.

(9) (Final 2010) A river running east-west is 6km wide. City A is located on the shore of the river; city B is located 8km to the east on the opposite bank. It costs \$40/km to build a bridge across the river, \$20/km to build a road along it. What is the cheapest way to construct a path between the cities?

Solution: (0) Picture



(1) Build a road of length x from A along the bank, then build a bridge of length d toward B.
(2) By Pythagoras, $d = \sqrt{6^2 + (8-x)^2}$.
(3) The total cost is

$$C(x) = 20x + 40\sqrt{6^2 + (8-x)^2} = 20x + 40\sqrt{6^2 + (x-8)^2}\,.$$

(4) The function $C(x)$ is defined everywhere ($6^2 + (8-x)^2 \geq 6^2 > 0$) and continuous there. We have

$$C'(x) = 20 + 40\frac{2(x-8)}{2\sqrt{6^2 + (x-8)^2}}\,.$$

This exists everywhere (the denominator is everywhere positive by the same calculation). It's enough to consider $0 \leq x \leq 8$ (no point in starting the bridge west of A or east of B). Looking for critical points we solve $C'(x) = 0$ that is:

$$\begin{aligned}
20 + 40\frac{x-8}{\sqrt{36 + (x-8)^2}} &= 0 \\
20 &= 40\frac{8-x}{\sqrt{36 + (8-x)^2}} \\
\sqrt{36 + (8-x)^2} &= 2(8-x) \\
36 + (8-x)^2 &= 4(8-x)^2 \\
36 &= 3(8-x)^2 \\
(8-x) &= \sqrt{\frac{36}{3}} = \sqrt{12} = 2\sqrt{3}
\end{aligned}$$

(only the positive root since $0 \leq x \leq 8$ forces $8 - x \geq 0$) so

$$x = 8 - 2\sqrt{3}\,.$$

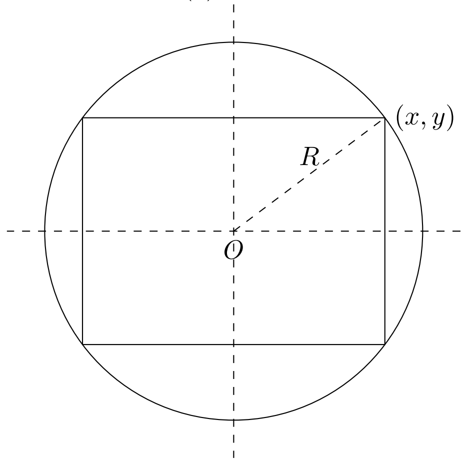
We then have $C(0) = 40\sqrt{6^2 + 8^2} = 40\sqrt{100} = 400$, $C(8) = 20 \cdot 8 + 40\sqrt{6^2} = 160 + 240 = 400$ and

$$\begin{aligned}
C(8 - 2\sqrt{3}) &= 20\left(8 - 2\sqrt{3}\right) + 40\sqrt{6^2 + (2\sqrt{3})^2} = 160 - 40\sqrt{3} + 40\sqrt{36 + 12} \\
&= 160 - 40\sqrt{3} + 40\sqrt{48} = 160 - 40\sqrt{3} + 40\sqrt{16 \cdot 3} \\
&= 160 - 40\sqrt{3} + 40 \cdot 4\sqrt{3} = 160 + 120\sqrt{3}\,.
\end{aligned}$$

Now $\sqrt{3} < \sqrt{4} = 2$ so $C(8 - 2\sqrt{3}) = 160 + 120\sqrt{3} < 160 + 120 \cdot 2 = 400 = C(0) = C(8)$ and we conclude that $C(8 - 2\sqrt{3})$ is the minimum.
(5) The cheapest way to construct a bridge is construct a road of length $\left(8 - 2\sqrt{3}\right)$ km along the bank from A toward B, and then bridge from the end of the road to B.
(6) Sanity checks: $0 < 2\sqrt{3} < 2 \cdot 2 < 8$ so the indeed the bridge starts somewhere betwen the cities. Our answer is on the kilometer scale.

(10) (Final 2019) Among all rectangles inscribed in a given circle, which one has the largest perimeter? Prove your answer.

Solution: (0) Picture



(1) We rotate the rectangle so that it's aligned with the axes; suppose one corner is at (x, y). Call the radius of the circle R and the perimeter of the rectangle P.
(2) We have $x^2 + y^2 = R^2$ so $y = \sqrt{R^2 - x^2}$.
(3) The total perimeter is

$$P(x) = 2x + 2y + 2x + 2y = 4(x + y) = 4(x + \sqrt{R^2 - x^2})$$

where $0 \le x \le R$.
(4) The function P is defined and continuous on $[0, R]$. We have

$$P'(x) = 4\left(1 - \frac{2x}{2\sqrt{R^2 - x^2}}\right)$$

This exists everywhere except at the endpoint $x = R$ where the denominator vanishes. There are critical points where $C'(x) = 0$ that is where

$$\begin{aligned} 4\left(1 - \frac{x}{\sqrt{R^2 - x^2}}\right) &= 0 \\ 1 &= \frac{x}{\sqrt{R^2 - x^2}} \\ \sqrt{R^2 - x^2} &= x \\ R^2 - x^2 &= x^2 \\ 2x^2 &= R^2 \\ x &= \frac{1}{\sqrt{2}} R. \end{aligned}$$

We have $P(\frac{1}{\sqrt{2}}R) = 4\left(\frac{1}{\sqrt{2}}R + \sqrt{R^2 - \frac{1}{2}R^2}\right) = 4\left(\frac{2}{\sqrt{2}}R\right) = 4\sqrt{2}R$ while at the endpoints we have $P(0) = 4\left(0 + \sqrt{R^2}\right) = 4R$ and $P(R) = 4\left(R + \sqrt{0}\right) = 4R$. It follows that the largest perimeter occurs when $x = \frac{1}{\sqrt{2}}R$.
(5) This rectangle also has $y = \sqrt{R^2 - x^2} = \frac{1}{\sqrt{2}}R$ so the rectangle with the largest perimeter is the square.

3. Extra problems

(11) Let $f(x) = xe^{-\alpha x^2}$ for $\alpha > 0$. What is the maximum value of f on on $[0, \infty)$? Where is it attained?
Solution: We have $f(x) > 0$ for all $x > 0$. We have $\lim_{x\to\infty} f(x) = 0$ since the super-exponential decay of $e^{-\alpha x^2}$ will win over the power law growth of x. Since $f(0) = 0$ and f is continuous it will have a maximum at some x in the interior, and since it is differentiable the maximum will be at a

critical point. Now

$$f'(x) = e^{-\alpha x^2}\left(1 - 2\alpha x^2\right),$$

so the only critical point is at $x = \frac{1}{\sqrt{2\alpha}}$ (recall that the domain is $x \geq 0$). The maximum value is then $\frac{1}{\sqrt{2\alpha e}}$.

(12) Owners of a car rental company have determined that if they charge customers d dollars per day to rent a car, the number of cars N they rent per day can be modelled by the function $N(d) = A - Bd$ where $A, B > 0$ are constants.

(a) What is the range of d for which this model makes sense?
Solution: The price should be positive, and the number of cars rented should be positive too, so we need $0 \leq d \leq \frac{A}{B}$.
Takeaway: Can sometimes determine the "sensible" range of the problem from the expressions.

(b) What price should they set to maximize their daily *revenue*?
Solution: The revenue for renting $N(d)$ cars at d dollars per day is $R(d) = N(d) \cdot d = (A - Bd)d = Ad - Bd^2$. This function is differentiable on $\left[0, \frac{A}{B}\right]$ were we have $R(0) = 0$ (if we don't charge rent we don't make money) and $R(\frac{A}{B}) = 0$ (if we rent no cars we don't make money). In between we have $R'(d) = A - 2Bd$ which vanishes at $d = \frac{A}{2B}$. Since f is positive in between the endpoints the maximum must be *somewhere* in the interval, and since there is only one critical point it must be the maximum, so the recommended number of cars is about $\frac{A}{2B}$. Alternative: evaluate $f\left(\frac{A}{2B}\right) = A \cdot \frac{A}{2B} - B \cdot \frac{A^2}{4B^2} = \frac{A^2}{4B} > 0$.
Takeaway: Can sometimes determine the "sensible" range of the problem from the expressions.
Solution: We have $R(d) = Ad - Bd^2 = \frac{A^2}{4B} - B\left(d - \frac{A}{2B}\right)^2$ so $R(d) \leq \frac{A^2}{4B}$ for all d and we achieve equality when $d = \frac{A}{2B}$ exactly.
Takeaway: Can sometimes use algebra without any calculus.

(13) A car factory can produce up to 120 units per week. Find the (whole number) quantity q of units which maximizes *profit* if the total revenue in dollars is $R(q) = (750 - 3q)q$, the total cost in dollars is $C(q) = 10,000 + 148q$ (observe the combination of *fixed* and *variable* costs).
Solution: The profits are the revenues minus the costs, so we need to maximize

$$\begin{aligned} P(q) &= R(q) - C(q) \\ &= 750q - 3q^2 - 148q - 10,000 \\ &= 602q - 3q^2 - 10,000\,. \end{aligned}$$

This function is differentiable on the closed interval $[0, 120]$ where $P'(q) = 602 - 6q$ achieving its maximum on $q_0 = \frac{602}{6} = 100\frac{1}{3}$. This is not an integer, so guess we need to round q_0 either right or left − but we need further analysis to decide which way. We look at the shape of the graph: $P'(q) > 0$ if $q < q_0$ and $P'(q) < 0$ if $q > q_0$ so the function is increasing on $\left[0, 100\frac{1}{3}\right]$ and decreasing on $\left[100\frac{1}{3}, 120\right]$. In particular the largest value on $[0, 100]$ is at 100 and the largest value on $[101, 120]$ is at 101. Using a calculator we find $P(100) = 22,200$ and $P(101) = 22,199$ so the best choice is to make 100 cars per week.
Takeaway: Can use the shape of the graph to round solutions (but watch out).