Math 100A – WORKSHEET 9
OPTIMIZATION

1. Optimization of functions

(1) Let $f(x) = x^4 - 4x^2 + 4$.
   (a) Find the absolute minimum and maximum of f on the interval $[-5, 5]$.

   (b) Find the absolute minimum and maximum of f on the interval $[-1, 1]$.

   (c) Find the absolute minimum and maximum of f (if they exist) on the interval $(-1, 1)$.

   (d) Find the absolute minimum and maximum of f (if they exist) on the real line.

(2) Let $f(x) = |x|$. Find the absolute minimum and maximum of f on the interval $[-1, 3]$.

(3) Find the global extrema (if any) of $f(x) = \frac{1}{x}$ on the intervals $(0, 5)$ and $[1, 4]$.

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2. Optimization problems

Problem-solving steps: (0) read carefully, draw picture; (1) fix coordinate system, name variables; (2) enforce relations; (3) create objective function; (4) calculus; (5) endgame; (6) sanity checks.

(4) A fish swimming at speed v relative to the water faces a drag force of the form av^2 and thus has to output a power of av^3. If the fish is swimming against a current of speed $u > 0$ (thus with speed $v > u$), it will cover a distance L at time $\frac{L}{v-u}$. The total energy cost is then $E = av^3 \frac{L}{v-u}$. At what speed v should the fish swim to minimize this cost?

(5) A standard model for the interaction between two neutral molecules is the *Lennard-Jones Potential* $V(r) = \epsilon \left[\left(\frac{r}{R}\right)^{-12} - 2\left(\frac{r}{R}\right)^{-6} \right]$. Here r is the distance between the molecules and $R, \epsilon > 0$ are parameters.

(a) What is the range of r values that makes sense?

(b) Physical systems tend to settle into a state of least energy. Find the minimum of this potential.

(6) Suppose we have 100m of fencing to enlose a rectangular area against a long, straight wall. What is the largest area we can enclose?

(7) A ferry operator is trying to optimize profits. Before each ferry trip workers spend some time loading cars after which the trip takes 1 hour. The ferry can carry up to 100 cars, each paying \$50 for the trip. Worker salaries total \$500/hour and the fuel for the trip costs \$250. The workers can load $N(t) = 100\frac{t}{t+1}$ cars in t hours.

(a) How much time should be devoted to loading to maximize profits *per trip*.

(b) The ferry runs continuously. How much time should be devoted to loading to maximize profits *per hour*?

(8) (Final 2012) The right-angled triangle ΔABP has the vertex $A = (-1, 0)$, a vertex P on the semicircle $y = \sqrt{1 - x^2}$, and another vertex B on the x-axis with the right angle at B. What is the largest possible area of such a triangle?

(9) (Final 2010) A river running east-west is 6km wide. City A is located on the shore of the river; city B is located 8km to the east on the opposite bank. It costs \$40/km to build a bridge across the river, \$20/km to build a road along it. What is the cheapest way to construct a path between the cities?

(10) (Final 2019) Among all rectangles inscribed in a given circle, which one has the largest perimeter? Prove your answer.

3. Extra problems

(11) Let $f(x) = xe^{-\alpha x^2}$ for $\alpha > 0$. What is the maximum value of f on on $[0, \infty)$? Where is it attained?

(12) Owners of a car rental company have determined that if they charge customers d dollars per day to rent a car, the number of cars N they rent per day can be modelled by the function $N(d) = A - Bd$ where $A, B > 0$ are constants.

(a) What is the range of d for which this model makes sense?

(b) What price should they set to maximize their daily *revenue*?

(13) A car factory can produce up to 120 units per week. Find the (whole number) quantity q of units which maximizes *profit* if the total revenue in dollars is $R(q) = (750 - 3q)q$, the total cost in dollars is $C(q) = 10,000 + 148q$ (observe the combination of *fixed* and *variable* costs).