# Math 100A – SOLUTIONS TO WORKSHEET 10
# TAYLOR EXPANSION

## 1. Taylor expansion

(1) (Review) Use linear approximations to estimate:
  (a) $\log\frac{4}{3}$ and $\log\frac{2}{3}$. Combine the two for an estimate of $\log 2$.
  **Solution:** Let $f(x) = \log x$ so that $f'(x) = \frac{1}{x}$. Then $f(1) = 0$ and $f'(1) = 1$ so $f(1+\frac{1}{3}) \approx \frac{1}{3}$ and $f(1-\frac{1}{3}) \approx -\frac{1}{3}$. Then $\log 2 = \log\frac{4}{3}/\frac{2}{3} = \log\frac{4}{3} - \log\frac{2}{3} \approx \frac{2}{3}$.
  **Takeaway**: Straightforward linear approximation using $f(x) \approx f(a) + f'(a)(x-a)$.
  **Common error:** Writing $f(x) \approx f(a) + f'(x)(x-a)$ (here: $\log x \approx \frac{1}{x}(x-1)$ ).
  **Sanity check**: is the expression we wrote a *linear function*?
  (b) $\sin 0.1$ and $\cos 0.1$.
  **Solution:** Let $f(x) = \sin x$ so that $g(x) = f'(x) = \cos x$ and $g'(x) = -\sin x$. Then $f(1) = 0$ and $g(0) = f'(0) = \cos 0 = 1$ while $g'(0) = -\sin 0 = 0$. So $f(0.1) \approx 0 + 1\cdot 0.1 \approx 0.1$ and $g(0.1) \approx 1 - 0\cdot 0.01 = 1$.
  **Takeaway**: Sometimes $f'(a) = 0$ and the linear approximation is constant.

(2) Let $f(x) = e^x$
  (a) Find $f(0), f'(0), f^{(2)}(0), \cdots$
  (b) Find a polynomial $T_0(x)$ such that $T_0(0) = f(0)$.
  (c) Find a polynomial $T_1(x)$ such that $T_1(0) = f(0)$ and $T_1'(0) = f'(0)$.
  (d) Find a polynomial $T_2(x)$ such that $T_2(0) = f(0)$, $T_2'(0) = f'(0)$ and $T_2^{(2)}(0) = f^{(2)}(0)$.
  (e) Find a polynomial $T_3(x)$ such that $T_3^{(k)}(0) = f^{(k)}(0)$ for $0 \le k \le 3$.
  **Solution:** $f(x) = f'(x) = f^{(2)}(x) = \cdots = e^x$ so $f(0) = f'(0) = f''(0) = \cdots = 1$. Now $T_0(x) = 1$ works, as does $T_1(x) = 1 + x$. If $T_2(x) = 1 + x + cx^2$ then $T_2''(x) = 2c = 1$ means $c = \frac{1}{2}$ and $T_2(x) = 1 + x + \frac{1}{2}x^2$. Finally, $T_3(x) = 1 + x + \frac{1}{2}x^2 + dx^3$ works if $6d = 1$ so if $d = \frac{1}{6}$.
  **Takeaway**: To determine coefficients of $x^2$, $x^3$ we needed to calculate with them without knowing their values, so we implement the problem-solving technique of *giving names*: by calling them $c, d$ we could convert the statements $T_2^{(2)}(0) = 1$ and $T_3^{(3)}(0) = 1$ into *equations* for $c, d$ which we could solve.

(3) Do the same with $f(x) = \log x$ about $x = 1$.
  **Solution:** $f'(x) = \frac{1}{x}$, $f''(x) = -\frac{1}{x^2}$, $f'''(x) = \frac{2}{x^3}$ so $f(1) = 0$, $f'(1) = 1$, $f''(1) = -1$, $f'''(1) = 2$. Try $T_3(x) = a + bx + cx^2 + dx^3$ (can truncate later). Need $a = 0$ to make $T_3(x) = 0$. Diff we get $T_3'(x) = b + 2cx + 3dx^2$, setting $x = 0$ gives $b = 1$. Diff again gives $T_3''(x) = 2c + 6dx$ so $2c = -1$ and $c = -\frac{1}{2}$. Diff again give $T_3'''(x) = 6d = 2$ so $d = \frac{1}{3}$ and $T_3(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$. Truncate this to get $T_0, T_1, T_2$.

Let $c_k = \frac{f^{(k)}(a)}{k!}$. The $n$th order Taylor expansion of $f(x)$ about $x = a$ is the polynomial

$$T_n(x) = c_0 + c_1(x-a) + \cdots + c_n(x-a)^n$$

(4) $\star$ Find the 4th order MacLaurin expansion of $\frac{1}{1-x}$ (=Taylor expansion about $x = 0$)
  **Solution:** $f'(x) = \frac{1}{(1-x)^2}$, $f''(x) = \frac{2}{(1-x)^3}$, $f^{(3)}(x) = \frac{6}{(1-x)^4}$, $f^{(4)}(x) = \frac{24}{(1-x)^5}$ $f^{(k)}(0) = k!$ and the Taylor expansion is $1 + x + x^2 + x^3 + x^4$.
  **Takeaway**: This is completely mechanical.

(5) Find the $n$th order MacLaurin expansion of $\cos x$, and approximate $\cos 0.1$ using the 3rd order expansion

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**Solution:** $(\cos x)' = -\sin x$, $(\cos x)^{(2)} = -\cos x$, $(\cos x)^{(3)} = \sin x$, $(\cos x)^{(4)}(x) = \cos x$ and the pattern repeats. Plugging in zero we see that the derivatives at 0 (starting with the zeroeth) are $1, 0, -1, 0, 1, 0, -1, 0, \ldots$ so the Taylor expansion is

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$$

In particular, $\cos 0.1 \approx 1 - \frac{1}{2}(0.1)^2 = 0.995$.

**Takeaway**: Again this is mechanical, but since the third derivative at $x = 0$ vanishes, we see that the third-order approximation actually only requires terms up to $x^2$, or equivalently that the quadratic approximation actually gains a free order of approximation.

(6) (Final, 2015) Let $T_3(x) = 24 + 6(x–3) + 12(x–3)^2 + 4(x–3)^3$ be the third-degree Taylor polynomial of some function $f$, expanded about $a = 3$. What is $f''(3)$?

**Solution:** We have $c_2 = \frac{f^{(2)}}{2!} = 12$ so $f^{(2)} = 24$.

**Takeaway**: We can use the formula $c_k = \frac{f^{(k)}(a)}{k!}$ both forwards (to go from $f$ to $c_k$) and *backwards* (to go from $c_k$ to $f^{(k)}(a)$).

(7) In special relativity we have the formula $E = \frac{mc^2}{\sqrt{1-v^2/c^2}}$ for the kinetic energy of a moving particle. Here $m$ is the "rest mass" of the particle and $c$ is the speed of light. Examine the behaviour of this formula for small velocities by expanding it to second order in the *small parameter* $x = v^2/c^2$. What is the 4th order expansion of the energy? Do you recognize any of the terms?

**Solution:** We write the formula as $E = mc^2(1-x)^{-1/2}$. Letting $f(x) = (1-x)^{-1/2}$ we have $f'(x) = \frac{1}{2}(1-x)^{-3/2}$ and $f''(x) = \frac{3}{4}(1-x)^{-5/2}$ so $f(0) = 1$, $f'(0) = \frac{1}{2}$ and $f''(0) = \frac{3}{4}$ giving the expansion

$$\begin{aligned} E &\approx mc^2\left(1 + \frac{1}{2}x + \frac{1}{2!}\cdot\frac{3}{4}x^2\right) \\ &= mc^2\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4}\right) \\ &= mc^2 + \frac{1}{2}mv^2 + \frac{3}{8}\left(\frac{v^2}{c^2}\right)mv^2 \end{aligned}$$

correct to 4th order in $v/c$. In particular we the famous *rest energy* $mc^2$ and that for small velocities the main contribution is the Newtonian kinetic energy $\frac{1}{2}mv^2$. The *first relativistic correction* is negative, and indeed is fairly small until $\frac{v}{c}$ gets close to 1.

**Takeaway**: Taylor expansion is a major workhorse of science.

## 2. New expansions from old

| | |
|---|---|
| Near $u = 0$: $\frac{1}{1-u} = 1 + u + u^2 + u^3 + u^4 \cdots$ | $\exp u = 1 + \frac{1}{1!}u + \frac{1}{2!}u^2 + \frac{1}{3!}u^3 + \frac{1}{4!}u^4 + \cdots$ |
| $\log(1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \frac{u^5}{5} - \cdots$ | |
| $\sin u = u - \frac{1}{3!}u^3 + \frac{1}{5!}u^5 - \frac{1}{7!}u^7 + \cdots$ | $\cos u = 1 - \frac{1}{2!}u^2 + \frac{1}{4!}u^4 - \frac{1}{6!}u^6 + \cdots$ |

(8) (Final, 2016) Use a 3rd order Taylor approximation to estimate $\sin 0.01$. Then find the 3rd order Taylor expansion of $(x+1)\sin x$ about $x = 0$.

**Solution:** To third order we have $\sin x \approx x - \frac{1}{6}x^3$. In particular $\sin 0.1 \approx 0.1 - \frac{1}{6000}$. We then also have, correct to third order, that

$$(x+1)\sin x \approx (x+1)\left(x - \frac{1}{6}x^3\right) = x + x^2 - \frac{1}{6}x^3 - \frac{1}{6}x^4 \approx x + x^2 - \frac{1}{6}x^3\,.$$

**Takeaway**: Rather than differentiate $(x+1)\sin x$ (which is doable but harder) we differentiated $\sin x$ by itself and then combined the resulting approximations. That $x^4$ is asymptotically negligible when we work to 3rd order was discussed in Lecture 1.

(9) Find the 3rd order Taylor expansion of $\sqrt{x} - \frac{1}{4}x$ about $x = 4$.

**Solution:** Let $f(x) = \sqrt{x}$. Then $f'(x) = \frac{1}{2\sqrt{x}}$, $f^{(2)}(x) = -\frac{1}{4x^{3/2}}$ and $f^{(3)}(x) = \frac{3}{8}x^{-5/2}$. Thus $f(4) = 2$, $f'(4) = \frac{1}{4}$, $f^{(2)}(4) = -\frac{1}{32}$, $f^{(3)}(4) = \frac{3}{256}$ and the third-order expansions are

$$\begin{aligned}
\sqrt{x} &\approx 2 + \frac{1}{4}(x-4) - \frac{1}{32\cdot 2!}(x-4)^2 + \frac{3}{256\cdot 3!}(x-4)^3 \\
\frac{1}{4}x &\approx 1 + \frac{1}{4}(x-4)
\end{aligned}$$

so that

$$\sqrt{x} - \frac{1}{4}x \approx 1 - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3 \,.$$

**Takeaway**: Here we added two expansions. We also *rebased* the polynomial $\frac{1}{4}x$ to be centered at $x = 4$.

(10) Find the 8th order expansion of $f(x) = e^{x^2} - \frac{1}{1+x^3}$. What is $f^{(6)}(0)$?

**Solution:** To fourth order we have $e^u \approx 1 + u + \frac{u^2}{2} + \frac{u^3}{6} + \frac{u^4}{24} + \frac{u^5}{120}$ so $e^{x^2} \approx 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \frac{x^8}{24}$ to 8th order. We also know that $\frac{1}{1-u} \approx 1 + u + u^2 + u^3$ so $\frac{1}{1+x^3} \approx 1 - x^3 + x^6$ correct to 8th order. We conclude that

$$\begin{aligned}
e^{x^2} - \frac{1}{1+x^3} &\approx \left(1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \frac{x^8}{24}\right) - \left(1 - x^3 + x^6\right) \\
&\approx x^2 - x^3 + \frac{1}{2}x^4 - \frac{5}{6}x^6 + \frac{1}{24}x^8 \,.
\end{aligned}$$

In particular, $\frac{f^{(6)}(0)}{6!} = -\frac{5}{6}$ so $f^{(6)}(0) = -720 \cdot \frac{5}{6} = -600$.

(11) Find the quartic expansion of $\frac{1}{\cos 3x}$ about $x = 0$.

**Solution:** To 4th order we have $\cos 3x \approx 1 - \frac{9}{2}x^2 + \frac{27}{8}x^4 = 1 - u$ where $u = \frac{9}{2}x^2 - \frac{27}{8}x^4$. Since $u^3$ is already a 6th order term we can truncate at the quadratic term of the geometric series:

$$\begin{aligned}
\frac{1}{\cos 3x} &\approx \frac{1}{1-u} \\
&\approx 1 + u + u^2 \\
&\approx 1 + \left(\frac{9}{2}x^2 - \frac{27}{8}x^4\right) + \left(\frac{9}{2}x^2 - \frac{27}{8}x^4\right)^2 \\
&\approx 1 + \frac{9}{2}x^2 - \frac{27}{8}x^4 + \frac{81}{4}x^4 \\
&= 1 + \frac{9}{2}x^2 + \frac{135}{8}x^4 \,.
\end{aligned}$$

correct to 4th order.

(12) (Change of variable/rebasing polynomials)

(a) Find the Taylor expansion of the polynomial $x^3 - x$ about $a = 1$ using the identity $x = 1 + (x-1)$.

**Solution:** We have

$$\begin{aligned}
x^3 - x &= (1 + (x-1))^3 - (1 + (x-1)) \\
&= 1 + 3(x-1) + 3(x-1)^2 + (x-1)^3 - 1 - (x-1) \\
&= 2(x-1) + 3(x-1)^2 + (x-1)^3 \,.
\end{aligned}$$

(b) Expand $e^{x^3-x}$ to third order about $a = 1$.

**Solution:** By the previous problem we have

$$\begin{aligned}\exp(x^3-x) &= \exp\left(2(x-1)+3(x-1)^2+(x-1)^3\right)\\ &\approx 1+\left(2(x-1)+3(x-1)^2+(x-1)^3\right)\\ &+\frac{1}{2}\left(2(x-1)+3(x-1)^2+(x-1)^3\right)^2\\ &+\frac{1}{6}\left(2(x-1)+3(x-1)^2+(x-1)^3\right)^3\end{aligned}$$

(no need to consider higher order terms because $u=2(x-1)+3(x-1)^2+(x-1)^3$ is a multiple of $(x-1)$ so any part of the $k$th power of $u$ has at least $k$th order in $(x-1)$. Expanding the powers and retaining only terms up to third order we get

$$\begin{aligned}\exp\left(x^3-x\right) &\approx 1+\left(2(x-1)+3(x-1)^2+(x-1)^3\right)\\ &+\frac{1}{2}\left(4(x-1)^2+12(x-1)^3\right)+\frac{1}{6}\left(8(x-1)^3\right)\\ &= 1+2(x-1)+5(x-1)^2+8\frac{1}{3}(x-1)^3\end{aligned}$$

correct to third order.

(13) Expand $\exp(\cos 2x)$ to sixth order about $x=0$.

**Solution:** We already know that $\cos\theta \approx 1-\frac{\theta^2}{2}+\frac{\theta^4}{24}-\frac{\theta^6}{720}$ correct to sixth order. Setting $\theta=2x$ we get

$$\begin{aligned}\exp(\cos 2x) &\approx \exp\left(1-2\theta^2+\frac{2}{3}\theta^4-\frac{4}{45}\theta^6\right)\\ &= e\cdot\exp\left(-2\theta^2+\frac{2}{3}\theta^4-\frac{4}{45}\theta^6\right)\\ &\approx e\left[1+\left(-2\theta^2+\frac{2}{3}\theta^4-\frac{4}{45}\theta^6\right)+\frac{1}{2}\left(-2\theta^2+\frac{2}{3}\theta^4-\frac{4}{45}\theta^6\right)^2+\frac{1}{6}\left(-2\theta^2+\frac{2}{3}\theta^4-\frac{4}{45}\theta^6\right)^3\right]\\ &= e\left[1-2\theta^2+\frac{2}{3}\theta^4-\frac{4}{45}\theta^6+\frac{1}{2}\left(4\theta^4-\frac{8}{3}\theta^6\right)-\frac{8}{6}\theta^6\right]\\ &= e\left[1-2\theta^2+2\frac{2}{3}\theta^4-\frac{124}{45}\theta^6\right]\\ &= e-2e\cdot\theta^2+\frac{8e}{3}\theta^4-\frac{124e}{45}\theta^6\,,\end{aligned}$$

correct to sixth order.

(14) Show that $\log\frac{1+x}{1-x}\approx 2(x+\frac{x^3}{3}+\frac{x^5}{5}+\cdots)$. Use this to get a good approximation to $\log 3$ via a careful choice of $x$.

**Solution:** Let $f(x)=\log(1+x)$. Then $f'(x)=\frac{1}{1+x}$, $f^{(2)}(x)=-\frac{1}{(1+x)^2}$, $f^{(3)}(x)=\frac{1\cdot 2}{(1+x)^3}$, $f^{(4)}(x)=-\frac{1\cdot 2\cdot 3}{(1+x)^4}$ and so on, so $f^{(k)}(x)=(-1)^{k-1}\cdot\frac{(k-1)!}{(1+x)^k}$. We thus have that $f(0)=0$ and for $k\geq 1$ that $f^{(k)}(0)=(-1)^{k-1}(k-1)!$ and $\frac{f^{(k)}(0)}{k!}=\frac{(-1)^{k-1}}{k}$. We conclude that

$$\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots.$$

Plugging $-x$ we get:

$$\log(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}\cdots$$

so

$$\log\frac{1+x}{1-x}=\log(1+x)-\log(1-x)=2x+2\frac{x^3}{3}+2\frac{x^5}{5}+\cdots.$$

In particular

$$\log 3 = \log \frac{1+\frac{1}{2}}{1-\frac{1}{2}} = 2\left(\frac{1}{2} + \frac{1}{24} + \frac{1}{160} + \cdots\right) = 1 + \frac{1}{12} + \frac{1}{80} + \cdots \approx 1.096$$

(15) (2023 Piazza @389) Find the asymptotics as $x \to \infty$

(a) $\sqrt{x^4 + 3x^3} - x^2$

**Solution:** Clearly as $x \to \infty$ $\sqrt{x^4 + 3x^3} \sim \sqrt{x^4} \sim x^2$ so this is about the cancellation and we need a more precise answer. Extracting the factor of $x^2$ from the square root we see

$$\sqrt{x^4 + 3x^3} - x^2 = x^2\sqrt{1 + \frac{3}{x}} - x^2 = x^2\left(\sqrt{1 + \frac{3}{x}} - 1\right).$$

To understand the behaviour of $\sqrt{1 + \frac{3}{x}} - 1$ we notice that $\frac{3}{x}$ is a *small parameter*, and that $\sqrt{1+u} \approx 1 + \frac{1}{2}u - \frac{1}{8}u^2$ correct to second order. We thus have

$$\begin{aligned}\sqrt{x^4 + 3x^3} - x^2 &\approx x^2\left(1 + \frac{1}{2}\frac{3}{x} - \frac{1}{8}\frac{9}{x^2} - 1\right)\\ &\approx \frac{3}{2}x - \frac{9}{8}\end{aligned}$$

with further corrections being lower order. We conclude that this linear approximation would have been sufficient and that

$$\sqrt{x^4 + 3x^3} - x^2 \sim \frac{3}{2}x$$

as $x \to \infty$.

(b) $\sqrt[3]{x^6 - x^4} - \sqrt{x^4 - \frac{2}{3}x^2}$

**Solution:** Both roots are asymptotically $x^2$. Using the linear approximation we find

$$\sqrt[3]{x^6 - x^4} = x^2\sqrt[3]{1 - \frac{1}{x^2}} \approx x^2\left(1 - \frac{1}{3}\frac{1}{x^2}\right)$$

and

$$\sqrt{x^4 - \frac{2}{3}x^2} \approx x^2\left(1 - \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{1}{x^2}\right)$$

which cancel exactly, so we need to go one order further. Since $(1+u)^\alpha \approx 1 + \alpha u + \frac{\alpha(\alpha-1)}{2}u^2 + \cdots$ as we can check by differentiation we see that as $x \to \infty$

$$\begin{aligned}\sqrt{1+u} &\approx 1 + \frac{1}{2}u - \frac{1}{8}u^2\\ \sqrt[3]{1+u} &\approx 1 + \frac{1}{3}u - \frac{1}{9}u^2\end{aligned}$$

to second order, so

$$\begin{aligned}\sqrt[3]{x^6 - x^4} - \sqrt{x^4 - \frac{2}{3}x^2} &\approx x^2\left[\left(1 - \frac{1}{3x^2} - \frac{1}{9x^4}\right) - \left(1 - \frac{1}{2}\frac{2}{3x^2} - \frac{4}{8\cdot 9x^4}\right)\right]\\ &\approx -\frac{1}{18x^2}\end{aligned}$$

with further lower-order terms, so

$$\sqrt[3]{x^6 - x^4} - \sqrt{x^4 - \frac{2}{3}x^2} \sim -\frac{1}{18x^2}$$

as $x \to \infty$ and in particular there is decay.

(16) Evaluate $\lim_{x\to 0} \frac{e^{-x^2/2}-\cos x}{x^4}$.

**Solution:** We know that $\cos x = 1 - \frac{x^2}{2} + \cdots$. Using the linear expansion $e^u \approx 1+u$ we'd get $e^{-x^2/2} \approx 1 - x^2/2$ which means the difference cancels to third order, so let's expand to fourth order. We get

$$e^{-x^2/2} \approx 1 - \frac{x^2}{2} + \frac{1}{2}\left(\frac{x^2}{2}\right)^2 = 1 - \frac{x^2}{2} + \frac{x^4}{8}$$
$$\cos x \approx 1 - \frac{x^2}{2} + \frac{x^4}{24}\,.$$

Subtracting and dividing by $x^4$ we get

$$\frac{e^{-x^2/2} - \cos x}{x^4} = \frac{1}{12}$$

correct to 0th order, so this is the limit (expanding both functions to the next order would give the next correction).