Math 100A – WORKSHEET 10
TAYLOR EXPANSION

1. Taylor expansion

(1) (Review) Use linear approximations to estimate:
  (a) $\log \frac{4}{3}$ and $\log \frac{2}{3}$. Combine the two for an estimate of $\log 2$.
  (b) $\sin 0.1$ and $\cos 0.1$.

(2) Let $f(x) = e^x$
  (a) Find $f(0), f'(0), f^{(2)}(0), \cdots$
  (b) Find a polynomial $T_0(x)$ such that $T_0(0) = f(0)$.
  (c) Find a polynomial $T_1(x)$ such that $T_1(0) = f(0)$ and $T_1'(0) = f'(0)$.
  (d) Find a polynomial $T_2(x)$ such that $T_2(0) = f(0)$, $T_2'(0) = f'(0)$ and $T_2^{(2)}(0) = f^{(2)}(0)$.
  (e) Find a polynomial $T_3(x)$ such that $T_3^{(k)}(0) = f^{(k)}(0)$ for $0 \le k \le 3$.

(3) Do the same with $f(x) = \log x$ about $x = 1$.

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Let $c_k = \frac{f^{(k)}(a)}{k!}$. The nth order Taylor expansion of $f(x)$ about $x = a$ is the polynomial

$$T_n(x) = c_0 + c_1(x-a) + \cdots + c_n(x-a)^n$$

(4) $\star$ Find the 4th order MacLaurin expansion of $\frac{1}{1-x}$ (=Taylor expansion about $x = 0$)

(5) Find the nth order MacLaurin expansion of $\cos x$, and approximate $\cos 0.1$ using the 3rd order expansion

(6) (Final, 2015) Let $T_3(x) = 24 + 6(x-3) + 12(x-3)^2 + 4(x-3)^3$ be the third-degree Taylor polynomial of some function f, expanded about $a = 3$. What is $f''(3)$?

(7) In special relativity we have the formula $E = \frac{mc^2}{\sqrt{1-v^2/c^2}}$ for the kinetic energy of a moving particle. Here m is the "rest mass" of the particle and c is the speed of light. Examine the behaviour of this formula for small velocities by expanding it to second order in the *small parameter* $x = v^2/c^2$. What is the 4th order expansion of the energy? Do you recognize any of the terms?

2. New expansions from old

Near $u=0$: $\frac{1}{1-u} = 1 + u + u^2 + u^3 + u^4 \cdots$	$\exp u = 1 + \frac{1}{1!}u + \frac{1}{2!}u^2 + \frac{1}{3!}u^3 + \frac{1}{4!}u^4 + \cdots$
$\log(1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \frac{u^5}{5} - \cdots$	
$\sin u = u - \frac{1}{3!}u^3 + \frac{1}{5!}u^5 - \frac{1}{7!}u^7 + \cdots$	$\cos u = 1 - \frac{1}{2!}u^2 + \frac{1}{4!}u^4 - \frac{1}{6!}u^6 + \cdots$

(8) (Final, 2016) Use a 3rd order Taylor approximation to estimate $\sin 0.01$. Then find the 3rd order Taylor expansion of $(x+1)\sin x$ about $x=0$.

(9) Find the 3rd order Taylor expansion of $\sqrt{x} - \frac{1}{4}x$ about $x=4$.

(10) Find the 8th order expansion of $f(x) = e^{x^2} - \frac{1}{1+x^3}$. What is $f^{(6)}(0)$?

(11) Find the quartic expansion of $\frac{1}{\cos 3x}$ about $x=0$.

(12) (Change of variable/rebasing polynomials)

(a) Find the Taylor expansion of the polynomial $x^3 - x$ about $a=1$ using the identity $x = 1 + (x-1)$.

(b) Expand e^{x^3-x} to third order about $a = 1$.

(13) Expand $\exp(\cos 2x)$ to sixth order about $x = 0$.

(14) Show that $\log \frac{1+x}{1-x} \approx 2(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots)$. Use this to get a good approximation to $\log 3$ via a careful choice of x.

(15) (2023 Piazza @389) Find the asymptotics as $x \to \infty$

(a) $\sqrt{x^4 + 3x^3} - x^2$

(b) $\sqrt[3]{x^6 - x^4} - \sqrt{x^4 - \frac{2}{3}x^2}$

(16) Evaluate $\lim_{x\to 0} \frac{e^{-x^2/2} - \cos x}{x^4}$.