Math 412, Lecture 9, 4/2/2025

Last time: tensor product $U \otimes V$.

Space spanned by "pure tensors" $\{\underline{u} \otimes \underline{v}\}_{\substack{u \in U \\ v \in V}}$ only subject to constraint that

$$\iota : U \times V \to U \otimes V$$

$$\iota(\underline{u}, \underline{v}) = \underline{u} \otimes \underline{v}$$

is bilinear.

Cor: If $B : U \times V \to Z$ is a bilinear map, there is a unique $\underbrace{f}_{\text{linear}} : U \otimes V \to Z$

s.t. $B = f \circ \iota$

Lemma: $B_U \subset U$, $B_V \subset V$ bases then $\left\{ \underline{u} \otimes \underline{v} \;\middle|\; \substack{\underline{u} \in B_U \\ \underline{v} \in B_V} \right\}$ is a basis of $U \otimes V$.

Examples:
(1) $\mathbb{R}^n \otimes_F \mathbb{R}^m \cong M_{nm}(\mathbb{R})$
(2) $U \otimes_F \{0\} \cong \{0\}$
(3) $U \otimes_F F \cong U$
(4) $F[x] \otimes_F F[y] \cong F[x,y]$

(morally: $\left(\substack{\text{functions on} \\ X}\right) \otimes \left(\substack{\text{functions} \\ \text{on } Y}\right) \cong \left(\substack{\text{functions} \\ \text{on } X \times Y}\right)$)

(5) QM: $\mathcal{U}, V$ state spaces of quantum systems,
$\mathcal{U} \otimes V$ " " of combined system

Observe: If $\dim_F \mathcal{U}, \dim_F V \geq 2$, have $t \in \mathcal{U} \otimes V$
not in image of τ (not pure tensors)

(QM: "entanglement")

Today: (1) Universal property,
(2) Extension of scalars

Def: An abstract tensor product of U, V is a pair (W, τ) where W is a vsp, $\tau: U \times V \to W$ is bilinear, and $\forall B: U \times V \to Z$ bilinear $\exists! f: W \to Z$ linear, s.t. $B = f \circ \tau$.

Example: $\operatorname{Im}(\tau)$ spans W. Else let $f: W \to F$ vanish on $\operatorname{Im}(\tau)$ (any nonzero functional on $W/\operatorname{Im}(\tau)$ will do)

Then $f \circ \tau = 0 \circ \tau$ so $f = 0$, so $W/\operatorname{Im}(\tau) = 0$
so $W = \operatorname{Im}(\tau)$

Recall Defi: $\{V_i\}_{i\in I}$ vsp? An abstract direct sum is a vsp W equipped with maps $\imath_i : V_i \to W$ st. For every system of maps $f_i \in \mathrm{Hom}_F(V_i, Z)$ $\exists!$ $f \in \mathrm{Hom}_F(W, Z)$ st. $f_i = f \circ \imath_i$.

Defi Same input: An abstract direct product is a vsp W, equipped with maps $\pi_i \in \mathrm{Hom}_F(W, V_i)$ st. for every system of maps $f_i \in \mathrm{Hom}_F(Z, V_i)$ $\exists!$ $f \in \mathrm{Hom}_F(Z, W)$ st $f_i = \pi_i \circ f$

Prop: each notion is unique up to unique isom.

Pf: Say $(W, \{\imath_i\})$, $(W', \{\imath_i'\})$ are direct sums.

Then $\{\imath_i' : V_i \to W'\}$ is a system of maps
$\{\imath_i : V_i \to W\}$ " " " " "

So $\exists f' : W \to W'$ st. $\imath_i' = f' \circ \imath_i$
$\exists f : W' \to W$ st. $\imath_i = f \circ \imath_i'$

So $f \circ f' : W \to W$ has $f \circ f' \circ \imath_i = \imath_i = id_W \circ \imath_i$
$f' \circ f : W' \to W'$ " $f' \circ f \circ \imath_i' = \imath_i' = id_{W'} \circ \imath_i'$

By uniqueness for maps into W, W' $f \circ f' = id_W$, $f' \circ f = id_{W'}$

Say (W, ι), (W', ι') are tensor pdts of U, V

then $\iota' : U \times V \to W'$ is bilinear

$\iota : U \times V \to W$ " "

by property of (W, ι) $\exists f' : W \to W'$: $f' \circ \iota = \iota'$

" " " (W', ι') $\exists f : W' \to W$: $f \circ \iota' = \iota$

so $f \circ f' \circ \iota = \iota = id_W \circ \iota$

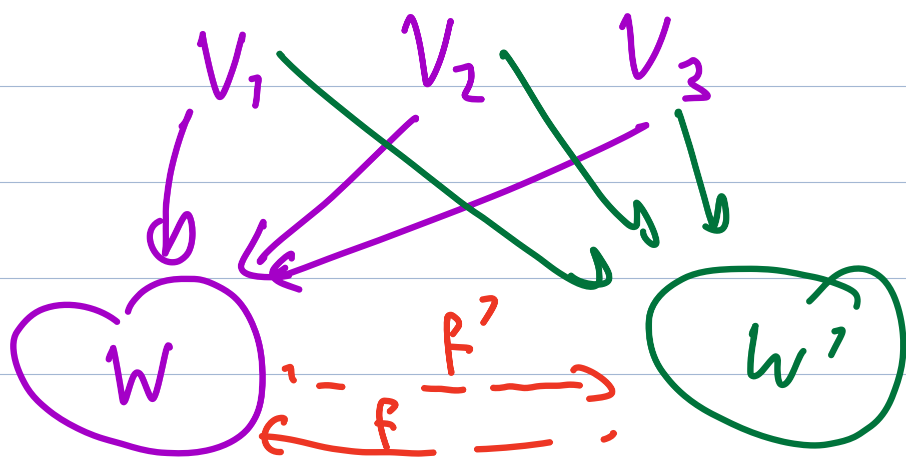
$f' \circ f \circ \iota' = \iota' = id_{W'} \circ \iota'$

By uniqueness for bilinear maps into W, W'

$f \circ f' = id_W$, $f' \circ f = id_{W'}$,

so f, f' are inverse ∎

Picture:



Remark: This point of view leads to Category theory. Forget about algebraic structures, only remember hom sets + composition law. Formulate existing notions purely in terms of objects & homs (no elements)

Example: Ex: say $f \in \mathrm{Hom}_{\mathcal{F}}(U,V)$ is an epimorphism if: $\forall Z\ \forall g_1, g_2 \in \mathrm{Hom}_{\mathcal{F}}(V,Z)$
if $g_1 \circ f = g_2 \circ f$ then $g_1 = g_2$.
Ex: f is an epi iff f is surjective.

Advantage: Prove thms once & for all.
Get notions once & for all

Example of tensor products:
Extension of Scalars

Goal: Consider matrix $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ it defines a linear map $\mathbb{Q}^2 \to \mathbb{Q}^2$, $\mathbb{R}^2 \to \mathbb{R}^2$, $\mathbb{C}^2 \to \mathbb{C}^2$
(start with $T: \mathbb{Q}^2 \to \mathbb{Q}^2$, $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+2y \\ 3x+4y \end{pmatrix}$.
want to say "extend T to $\mathbb{R}^2, \mathbb{C}^2$ in obvious way".

What if we just set $T: V \to V$, V is a $\mathbb{Q}$-vsp?

Approach 1: Choose basis $B = \{v_i\}_{i \in I}$

Define $V_{\mathbb{C}} = \bigoplus_i \mathbb{C} v_i$ (formal span)

Def $T_{\mathbb{C}}$ by $T_{\mathbb{C}} \underline{v}_i = T \underline{v}_i$

say $T v_i = \sum_j a_{ji} v_j$

set $T_{\mathbb{C}} \underline{v}_i = \sum_j a_{ji} v_j \in V_{\mathbb{C}}$

Objection / Check: if we change basis, set "same thing".

Fix field F, let K/F be a field extension

PS 1: K has structure of F-vsp

Lemma-Definition: Let V be an F-vsp. Then $V_K \overset{\text{def}}{=} \underset{\text{as } F\text{-vsp}}{K} \otimes_F V$ has structure of a K-vsp

Pf: Given $\alpha \in K$ want to define $\alpha \underline{t}$, $\underline{t} \in K \otimes_F V$.
so $(\alpha \cdot): V_K \to V_K$ will be linear.

But this is equivalent to defining $(\alpha \cdot): K \times V \to V_K$ bilinear. Set

$$\alpha \cdot (a, \underline{v}) \overset{def}{=} \alpha a \otimes \underline{v}$$

If $\alpha = 1$, set $1 \cdot (a, \underline{v}) = a \otimes \underline{v}$ $\leftarrow$ just 2

so by uniqueness

$$(1 \cdot): V_K \to V_K \text{ is } id_{V_K}$$

Also $(\alpha \cdot) \cdot (a \otimes \underline{v}) = (\alpha a) \otimes \underline{v}$

$$(\alpha \beta) \cdot (\underline{a} \otimes v) = ((\alpha\beta) a) \otimes \underline{v} = (\alpha (\beta a)) \otimes \underline{v}$$

$$= (\alpha \cdot) ((\beta a) \otimes \underline{v}) = (\alpha \cdot)(\beta \cdot)(\underline{a} \otimes \underline{v})$$

pure tensors span, so $(\alpha\beta) \cdot = (\alpha \cdot)(\beta \cdot)$

Similarly: That $(\alpha \cdot)$ is $\overset{F}{\text{linear}}$ gives

$$(\alpha \cdot)(\underline{t}_1 + \underline{t}_2) = \alpha \cdot \underline{t}_1 + \alpha \cdot \underline{t}_2$$

Also

$$(\alpha + \beta) \cdot (a \otimes \underline{v}) \overset{\text{Constr}}{=} ((\alpha + \beta) a) \otimes \underline{v} \overset{K}{=} (\alpha a + \beta a) \otimes \underline{v}$$

$$\overset{\otimes}{=} (\alpha a) \otimes \underline{v} + (\beta a) \otimes \underline{v} = \alpha \cdot (a \otimes \underline{v}) + \beta \cdot (a \otimes \underline{v}).$$

$\blacksquare$

(remark: from F-linear maps $\begin{matrix} T_u : U \to U' \\ T_v : V \to V' \end{matrix}$ set

F-linear map $T_u \otimes T_v : U \otimes V \to U' \otimes V'$
satisfies $(T_u \otimes T_v)(\underline{u} \otimes \underline{v}) = T_u \underline{u} \otimes T_v \underline{v}$.
called Kronecker pdt)

matrices a_{ij}, b_{kl}, set $c_{ik,jl} = a_{ij} b_{kl}$)

Theorem: Let $B \subset V$ be an F-basis. Then $\{1 \otimes \underline{v} \mid \underline{v} \in B\}$ is a K-basis of V_K.

Pf: Spanning: know $\left\{ a \otimes \underline{v} : \begin{matrix} a \in K \\ \underline{v} \in B \end{matrix} \right\}$ spans V_K as an F-vsp. But K-span of $\{1 \otimes \underline{v}\}_{\underline{v} \in B}$ certainly contains this

Fix F-basis $C \subset K$. F-basis of V_K.

$\text{span}_K \{1 \otimes \underline{v} : \underline{v} \in B\} \supset \left\{ c \otimes \underline{v} \;\middle|\; \begin{matrix} \underline{v} \in B \\ c \in C \end{matrix} \right\}$

so

$\text{span}_K \{1 \otimes \underline{v} : \underline{v} \in B\} \supset \text{span}_F \{ \quad \} = V_K$

Concretely: $\sum_i \alpha_i (1 \otimes \underline{v}_i) = 1 \otimes \sum_i \alpha_i \underline{v}_i$ (bilinearity) $\quad \alpha_i \in F$

<u>Indep</u>: need to show: if $\sum_i \alpha_i (1 \otimes v_i) = 0$
$\alpha_i \in K$, $v_i \in B$ distinct, then $\alpha_i = 0$

$\iff$ $\sum_i \alpha_i \otimes v_i = 0$ then all $\alpha_i = 0$

To show this, can write each α_i as

$$\alpha_i = \sum_j \beta_{ij} c_j$$

$\{c_j\} \subset C$ distinct
$\beta_{ij} \in F$

Then set:

$$\sum_i \Big(\big(\sum_j \beta_{ij} c_j \big) \otimes v_i \Big) = 0$$

$$\sum_{i,j} \beta_{ij} \; c_j \otimes v_i = 0$$

distinct elements of F-basis

So all $\beta_{ij} = 0$, so each $\alpha_i = 0$ ∎

<u>Conclusion</u>: $\dim_K V_K = \dim_F V$,
V_K really is "K-span of basis of V"

informally: think of $K \otimes_F V$ as "all K-linear combos of vectors in V, subject to compatibility of F action on both sides:

$$\alpha a \otimes_F \underline{v} = a \otimes_F \alpha \underline{v} \quad \forall \ \alpha \in F$$

HW: Given $T: U \to V$ can make $T_K: U_K \to V_K$ ($T_K := 1_K \otimes T$) check: (1) matrix of T_K = matrix of T

(2) $(T \circ S)_K = T_K \circ S_K$ ("functoriality")

HW: All constructions (direct sum, direct pdt, quotient, dual space, tensor pdt) respect ext. of scalars. Also Ker, Im, ..

Eg. $U_K \otimes_K V_K \cong (U \otimes_F V)_K$