Math 412, Lecture 15

Last time: Fixed F-vsp V, $T \in \mathrm{End}_F(V)$

For $f \in F[x]$, defined $f(T) = \sum_{i=0}^{d} a_i T^i$

if $f = \sum_{i=0}^{d} a_i x^i$.

Saw: $(\alpha f + g)(T) = \alpha f(T) + g(T)$

$(fg)(T) = f(T)\,g(T)$

$\Rightarrow I = \{ f \mid f(T) = 0 \} \triangleleft F[x]$

I has a element m_T of minimal degree, then $I = (m_T) = \{ m_T f : f \in F[x] \}$.

m_T is unique called the minimal poly of T (If $\dim_F V < \infty$, $\deg m_T > 0$.)

Lemma: If $T\underline{v} = \lambda \underline{v}$, $f(T)\underline{v} = f(\lambda)\underline{v}$

(if $TW \subset W$ then $f(T)W \subset W$)

(W subspace)

$\Rightarrow$ If $T\underline{v} = \lambda \underline{v}$, $\underline{v} \neq 0$, $m_T(\lambda) = 0$

Prop: T invertible iff $m_T(0) \neq 0$

$\Rightarrow \mathrm{Spec}_F(T) =$ eigenvalues $=$ spectrum $= m_T^{-1}(0)$.

Def: Generalized eigenspace

$$V_\lambda = \{ \underline{v} \in V \mid \exists k : (T-\lambda)^k \underline{v} = \underline{0} \}$$

Saw: (1) V_λ is T-inv't

(2) $T-\mu \restriction_{V_\lambda}$ is injective if $\mu \neq \lambda$

(3) $V_\lambda \neq \{\underline{0}\}$ iff $\lambda \in \mathrm{Spec}_F(T)$

Today: Investigate generalized eigenspace, prove Cayley-Hamilton, .

Theorem: The sum $\sum_\lambda V_\lambda$ is direct

Pf: Let $\sum_{i=1}^{r} \underline{v}_i = \underline{0}$ be a minimal dependence,

$\underline{v}_i \in V_{\lambda_i}$, λ_i distinct. By minimality all $\underline{v}_i \neq \underline{0}$.

Apply $(T-\lambda_r)^k$, k large so that $(T-\lambda_r)^k \underline{v}_r = \underline{0}$.

Then $\sum_{i=1}^{r-1} (T-\lambda_r)^k \underline{v}_i = \underline{0}$

$(T-\lambda_r)^k v_i \in V_{\lambda_i}$ since V_{λ_i} is T-inv't,

They are non zero since $T-\lambda_r$ is injective on V_{λ_i}. (Here we use $\lambda_i \neq \lambda_r$ if $i \neq r$)

$\blacksquare$

<u>Remark:</u> $\mathrm{Spec}_F(T)$ can be empty.

E.g. $T = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \in M_2(\mathbb{R})$, $T^2 = -I$, x^2+1 has no real roots.

<u>Ex:</u> Strengthed the claim to show $T-\mu|_{V_\lambda}$ is invertible, if $\mu \neq \lambda$.

<u>Algebraically closed fields</u>

$\dim_F V = n < \infty$.

When $T \in \mathrm{End}_F(V)$ fails have eigenvalues, we blame F not on T.

<u>Def:</u> Call field E <u>algebraically closed</u> if every non-constant $f \in E[x]$ has a root in E.

$\iff$ Every non-zero polynomial is a product of linear polynomials.

Fact: (Fundamental Theorem of Algebra)
$\mathbb{C}$ is algebraically closed

This is a thm of analysis, different proofs, eg. : (1) if $f \in \mathbb{C}[x]$ is nowhere zero,

all analysis $\to$ $\frac{1}{f}$ is a bounded entire fcn hence constant.

(2) If $f \in \mathbb{R}[x]$ has odd degree then f has a real root.

If $f \in \mathbb{C}[x]$ is quadratic, f has a complex root.

+ Galois theory.

Goal: Have enough e.v. of T so that

$$V = \bigoplus_{\lambda} V_{\lambda}$$ ("spectral theorem")

Define problem away by extending scalars

Thm: Any field F is contained in an algebraically closed field; the minimal such field is unique, called the algebraic closure of F. Denoted $\bar{F}$.

Proof: If $f \in F[x]$ doesn't have a root in F, add the root by extending F

Process stops by Zorn's lemma.

This process of adding roots is the "closure" in "algebraically closed".

Do we need $\bar{F}$? A: no.
Enough to make a finite extension in which m_T splits.

Apply this to linear maps by extending scalars to $\bar{F}$.

(1) embed $M_n(F) \subset M_n(\bar{F})$

(2) Extend scalars to $T_{\bar{F}} = Id_{\bar{F}} \otimes_F T$
$\in End(\bar{F} \otimes_F V)$

HW: T_K has $m_{T_K} = m_T$, $p_{T_K} = p_T$

(relative to inclusion $F \subset K$)

Lemma: Suppose F is alg. closed, $1 \le \dim_F V < \infty$. Then every $T \in \mathrm{End}_F(V)$ has an eigenvalue.

Pf: m_T is a non-constant poly, so has a root in F.

Recall pf of Spectral thm for SA T:

(1) eigenspaces of T are $\perp$: $V_\lambda \perp V_\mu$
$\Rightarrow \bigoplus_\lambda V_\lambda$ is direct

(2) Let $W = \bigoplus_\lambda V_\lambda$; W inv't so $W^\perp$ is inv't.
If $W \neq V$, $W^\perp \neq \{0\}$, $T|_{W^\perp}$ has eigenvector
$\Rightarrow \Leftarrow$

Theorem: (F alg. closed, $\dim_F V < \infty$) then

$$V = \bigoplus_{\lambda \in \mathrm{Spec}_F(T)} V_\lambda.$$

Pf: Let $m_T(x) = \prod_{i=1}^{r} (x - \lambda_i)^{k_i}$.

let $W = \bigoplus_{\lambda} V_\lambda$. <u>Goal:</u> $W = V$

let $\bar{V} = V/W$ (want: $\bar{V} = \{\underline{0}\}$), consider quotient map $\bar{T} : \bar{V} \to \bar{V}$

$$\bar{T}(\underline{v} + W) = T\underline{v} + W.$$

suppose $W \neq V$.

$\bar{T} \in \mathrm{End}_F(\bar{V})$, $\quad 1 \leq \dim_F \bar{V} \leq \dim_F V < \infty$

$$\dim_F V - \dim_F W > 0.$$

so $\bar{T}$ has an eigenvector $\underline{v} + W \neq \underline{0}$ with e.v. λ.

(1) <u>claim</u>: $\lambda \in \mathrm{Spec}_F(T)$. For any $f \in F[x]$,

$$f(\bar{T}) = \overline{f(T)} \qquad \left(f(\bar{T})(\underline{v} + W) = (f(T)\underline{v}) + W\right)$$

$$\Rightarrow m_T(\bar{T}) = \overline{m_T(T)} = \underline{0}.$$

so $m_{\bar{T}} \mid m_T$ so all roots of $m_{\bar{T}}$ are among roots of m_T, so $\lambda \in \mathrm{Spec}_F(T)$.

So, renumbering e.v. may assume $\lambda = \lambda_r$.
Let $\underline{v} \in V$ s.t. $(\bar{T} - \lambda_r)(\underline{v} + W) = \underline{0}$

$$\iff (T - \lambda_r)\underline{v} \in W.$$

$p(x) = \prod_{i=1}^{r-1} (x - \lambda_i)^{k_i}$. Then $p(\bar{T})$ is invertible on $\bar{V}_{\lambda_r}$

So $\quad p(\bar{T})(\underline{v} + W) \neq \underline{0}_{\bar{V}}$

Let $\underline{u} = p(T)\underline{v}$. Then $\overline{p(T)\underline{u}} \neq \underline{0}_{V/W}$

$$\underline{u} = p(T)\underline{v} \notin W.$$

Then $(T - \lambda_r)^{k_r}\underline{u} = (T - \lambda_r)^{k_r} \cdot \prod_{i=1}^{r-1}(T - \lambda_i)^{k_i}\underline{u}$

$$= m_T(T)\underline{u} = \underline{0}$$

$\Rightarrow \underline{u} \in V_{\lambda_r}$, $\underline{u} \notin W$. Contradiction.

$\blacksquare$

Interpretation: wrt $\bigoplus_\lambda V_\lambda = V$, T is block-diagonal, enough to analyze $T|_{V_\lambda}$:
$T = \bigoplus_\lambda T|_{V_\lambda}$.

unlike SA spectral thm, no promise that $\tau|_{V_\lambda}$ is scalar, even though it has one e.v. λ

<u>Prop:</u> $m_{\tau|_{V_{\lambda_i}}}$ is $(x-\lambda_i)^{k_i}$, i.e. k_i is <u>minimal</u> so that $(\tau-\lambda_i)^{k_i}|_{V_{\lambda_i}} = 0$

(k_i is the degree of nilpotence of $\tau-\lambda_i$).

<u>Pf:</u> Since λ is only eigenvalue of τ in V_λ, min poly is $(x-\lambda)^k$ where k is minimal st.

$$(\tau|_{V_\lambda}-\lambda)^k = 0$$

Since $m_\tau(\tau|_{V_\lambda}) = m(\tau)|_{V_\lambda} = 0$,

$m_{\tau|_{V_\lambda}} \mid m_\tau$ so $k \le k_i$ if $\lambda = \lambda_i$.

Conversely, $p_i(\tau|_{V_\lambda}) \cdot (\tau|_{V_\lambda} - \lambda)^{k_i} = 0$

$$\overset{m_\tau(\tau|_{V_\lambda})}{\text{''}}$$

$p_i = \prod_{j \ne i} (x-\lambda_j)^{k_j}$, p_i invertible, so $(\tau|_{V_{\lambda_i}} - \lambda_i)^{k_i} = 0$

so $k_i \le k$.

<u>Conclusion:</u> m_T encodes the degree of nilpotence of each $T|_{V_\lambda}$.

- F alg closed, $n = \dim_F V < \infty$.
- $m_T = \prod_{i=1} (x - \lambda_i)^{k_i}$
- $(T|_{V_{\lambda_i}} - \lambda_i)^{k_i} = 0_{V_{\lambda_i}}$, $(T|_{V_{\lambda_i}} - \lambda_i)^{k_i - 1} \ne 0$.

Set $N = (T - \lambda)|_{V_\lambda}$. Then $N^k = 0$
so N is nilpotent.

<u>Lemma:</u> deg of nilpotence of $N \in \mathrm{End}_F(V_\lambda)$ is at most $\dim_F V_\lambda$.

Pf: HW.

<u>Cor:</u> (Cayley-Hamilton Theorem) suppose F is algebraically closed. Then $m_T \mid p_T$
$\Rightarrow p_T(T) = 0$

Recall $p_T(x) = \det(x\,Id_V - T)$.

Pf: The linear map $x-T$ respects the direct sum decomp $V = \bigoplus_\lambda V_\lambda$

so $\det\left(\bigoplus_\lambda (x-T)\restriction_{V_\lambda}\right) = \prod_\lambda (x - T\restriction_{V_\lambda}) =$

$$= \prod_\lambda p_{T\restriction_{V_\lambda}}(x)$$

So. $p_T = \prod_\lambda p_{T\restriction_{V_\lambda}}$

But λ ev. of T in V_λ iff λ root of $p_{T\restriction_{V_\lambda}}$ so λ only ev., so

$$p_{T\restriction_{V_\lambda}}(x) = (x-\lambda)^{\dim_F V_\lambda}$$

By lemma, $k_\lambda \leq \dim_F V_\lambda$

so $(x-\lambda_i)^{k_i} \mid (x-\lambda_i)^{\dim V_{\lambda_i}}$

mult over i, get

$$m_T \mid p_T.$$

Lemma: If E/K fields, $T \in \mathrm{End}_F(T)$,

$$m_{T_K} = m_T, \quad p_{T_K} = p_T$$

Theorem: (Cayley-Hamilton 2) $T \in \mathrm{End}_F(V)$ F any field $\Rightarrow p_T(T) = 0$.

Pf: $(p_T(T))_{\bar{F}} = p_{T_{\bar{F}}}(T_{\bar{F}}) = 0$

so $p_T(T) = 0$

Or: $\dfrac{p_{T_{\bar{F}}}}{m_{T_{\bar{F}}}} \in \bar{F}[x] \cap F(x) = F[x]$.

Remark: let $A = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \in M_n(\mathbb{Z}[\underline{x}])$

$\subseteq M_n(\mathbb{C}(\underline{x}))$

Apply C-H in $\mathbb{C}(\underline{x})$

to get $p_A(A) = 0$

observe: $p_A(t) = \mathbb{Z}[\underline{x}](t)$

so $p_A(A) = M_n(\mathbb{Z}[\underline{x}])$

Thus All entries of this matrix are 0, so entries are 0 poly.

$\Rightarrow$ identity $p_A(A) = 0$ is an identity in $M_n(\mathbb{Z}[\underline{x}])$

and so holds true if we evaluate it over any (commutative) ring.

R ring $\underline{a} \in R^m$, $f \in \mathbb{Z}[x]$

then $f \mapsto f(\underline{a})$ is an algebra hom $\mathbb{Z}[\underline{x}] \to R$

if $f = 0$, $f(\underline{a}) = \underline{0}$

$(x-y)^2 = x^2 - 2xy + y^2$

<u>Hamilton's proof</u>: Evaluate by hand

$\det\left(t \cdot I - \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \end{pmatrix}\right)$

as poly in $t, x_{11}, .., x_{1n}$ if $n=1,2,3$

Plug in $t = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \\ x_{n1} & \cdots & x_{nn} \end{pmatrix}$

Cancel, see you get 0.

Therefore for all n.

try $n=2$ by hand

Chevalley-Warning $F = \mathbb{F}_q$

$f \in \mathbb{F}_q[x_1, .., x_n]^{\leq d}$

if d small enough (relative to n, q)
f will have zeroes

Over $\mathbb{F}_q$ only have finitely many

functions $\mathbb{F}_q^n \to \mathbb{F}_q$, (have $q^{(q^n)}$)

$\Rightarrow$ map $\{\text{polynomials}\} \to \{\text{poly functions}\}$

is not injective (it is surjective)

poly $x^q - x \equiv 0$ on $\overline{\mathbb{F}}_q$.

$\overline{\mathbb{F}}_q$ is an field