Math 412, Lecture 16

Last time: $T \in \mathrm{End}_F(V)$, $\dim_F V < \infty$.

Generalized eigenspaces $V_\lambda = \{v \in V \mid \exists k: (T-\lambda)^k v = 0\}$

min poly $m_T(x) \in F[x]$

Thm: (Spectral thm): Suppose m_T splits in F (e.g. F algebraically closed). Then

$$V = \bigoplus_\lambda V_\lambda.$$

From this: (1) If $m_T(x) = \prod_{i=1}^{r} (x-\lambda_i)^{k_i}$

$p_T(x) = \prod_{i=1}^{r} (x-\lambda_i)^{n_i}$

Then (1) $T|_{V_{\lambda_i}} - \lambda_i$ is nilpotent of deg k_i

$\Rightarrow$ (2) $p_T(T) = 0$ so $m_T \mid p_T$.

Thm (Cayley-Hamilton).

Today: Choosing a good basis for a nilpotent map

Let $N \in \mathrm{End}_F(V)$ satisfy $N^k = 0$

(e.g. $N = T|_{V_\lambda} - \lambda$, $V = V_\lambda$)

Observe. $\{\underline{0}\} = \ker(N^0) \subset \ker(N)$

$$\subset \ker(N^2)$$
$$\vdots$$
$$\subset \ker(N^k)$$

In fact these are distinct.

If $\ker(N^{j+1}) = \ker(N^j)$

$\Rightarrow$ on image of N^j, N is injective.

(if $N(N^j \underline{v}) = \underline{0}$ then $N^j \underline{v} = \underline{0}$).

(1) Choose basis (size m_1) in $\ker(N^1)$

(2) Extend by m_2 vectors to get basis of $\ker(N^2)$

$\vdots$

(3) Extend by m_k vectors to get basis of $V = \ker(N^k)$

In this basis, matrix looks: block-matrix
blocks of size $m_i \times m_j$

if $N^j \underline{v} = \underline{0}$ then $N^{j-1}(N\underline{v}) = \underline{0}$

so N is a kind of shift here:
if $\underline{v} \in$ basis was chosen in jth step, $N\underline{v}$
is combo of vectors from previous steps

$$\begin{pmatrix} 0_{m_1} & m_1 \times m_2 & m_1 \times m_3 & \cdots \\ & 0_{m_2} & m_2 \times m_3 & \\ & & 0_{m_3} & \\ & 0 & & \ddots \end{pmatrix}$$

zero below diagonal

zeroes on block-diagonal

<u>Idea:</u> choose basis more carefully so N
shifts basis vectors, not just subspaces

Observe: need to choose vectors not from $\ker(N^k)$ s.t. $N^{k-1}\underline{v}$ is in basis of $\ker N$

but $V \setminus \ker(N^{k-1})$ not a subspace, can't just choose a "basis" there

Lemma: Let N be nilpotent. Let $B \subset V$ be a set of vectors such that: $N(B) \subset B \cup \{\underline{0}\}$

Then B is linearly indep iff $B \cap \ker(N)$ is.

Pf: let $\sum_{i=1}^{r} \alpha_i \underline{v}_i = \underline{0}$ be a minimal dependence in B. Let m be maximal s.t. $N^m \underline{v}_i \neq 0$ for some i ($\exists$ such m because N is nilpotent)

Then $\sum_{i=1}^{r} \alpha_i (N^m \underline{v}_i) = \underline{0}$

the $N^m \underline{v}_i \neq \underline{0}$ (else we get shorter dependence)
All in $\ker(N)$ (if not, $N^{m+1} \underline{v}_i \neq \underline{0}$ for some i.

$\Rightarrow$ just got a dependence in $B \cap \ker(N)$.

Warning: We never showed the vectors $N^{j}\underline{v}$ are distinct, & they don't have to be

Correct lemma: Assume $N(B) \subset B \setminus \{\underline{0}\}$
& N is injective on $B \setminus \ker(N)$.

Example: $\underline{v}$ be such that $N^k \underline{v} = 0$, $N^{k-1}\underline{v} \neq \underline{0}$

consider $B = \{\underline{v}, N\underline{v}, N^2\underline{v}, \dots, N^{k-1}\underline{v}\}$

indeed N shifts these vectors: $N(N^j\underline{v}) = N^{j+1}\underline{v}$.

injective:

each $N^j\underline{v} \in \ker(N^{k-j}) \setminus \ker(N^{k-j-1})$

so N injective here

$B \cap \ker(N) = \{N^{k-1}\underline{v}\}$ indep (vector is $\neq 0$)

Then $\mathrm{span}(B)$ is N-inv't, matrix of $N|_{\mathrm{span}(B)}$ wrt B is

(reverse order)

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \\ \vdots & \vdots & 0 & 1 & \\ & & \vdots & 0 & \ddots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \qquad a_{ij} = \begin{cases} 1 & j = i+1 \\ 0 & j \neq i+1 \end{cases}$$

matrix has 1s on the diagonal above main
0 elsewhere

Def: We call such a matrix a Jordan block (also such a subspace)

Thm: $N \in \mathrm{End}_F(V)$ nilp $\Rightarrow$ (1) V is a direct sum of Jordan blocks; (2) number of blocks of each size only depends on N.

Example: $A = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \in M_3(\mathbb{Q})$

$$A^2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \left[\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right] \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$$
$$= \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} (0) \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} = 0$$

so A is nilpotent of deg 2

$\mathrm{Im}(A) = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\} \subset \mathrm{Ker}(A)$ $(A^2 = 0)$

note: $A \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$. (solve linear equations)

then $\left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} \right\}$ is a Jordan block for A.

$\dim \mathrm{Ker}(A) = 3 - \dim \mathrm{Im}(A) = 2$,

choose $\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \in \mathrm{Ker}(A)$ indep of $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

Then $\left\{ \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \right\}$ is a Jordan block,

let $B = \left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \right\}$ then the matrix of A wrt B is

block of size 2 $\rightarrow$ $\begin{pmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} & 0 \\ 0 & (0) \end{pmatrix}$

block of size 1

can't have block of size 3 $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
(that's nilp of deg 3)

can't have 3 blocks of size 1 : $A \neq 0$

<u>Proof of thm</u>: Say N is nilp of deg d.

for $0 \le k \le d$ set $W_k = \mathrm{Im}(N^k) \cap \mathrm{Ker}(N)$

so $W_0 = W = \mathrm{Ker}(N) \supset W_1 \supset W_2 \supset \cdots \supset W_d = \{0\}$.

Choose basis C of W consistent with this decomp: choose basis $C_{d-1} \subset W_{d-1}$,
extend to basis $C_{d-1} \cup C_{d-2} \subset W_{d-2}$

$\vdots$

so $C_k \subset W_k$ s.t. $\bigcup_{k' \geq k} C_{k'}$ set basis of W_k

let $C = \bigcup_k C_k = \{ \underline{v}_i \}_{i \in I}$ define k_i s.t. $\underline{v}_i \in C_{k_i}$.

Now each $\underline{v}_i \in \operatorname{Im}(N^{k_i}) \cap W$, so $\exists \underline{u}_i \in V$ s.t. $N^{k_i} \underline{u}_i = \underline{v}$.

For $i \in I$, $0 \leq j \leq k_i$ set $\underline{v}_{i,j} = N^{k_i - j} \underline{u}_i$

so $\underline{v}_{i,0} = \underline{v}_i$, $N \underline{v}_{i,1} = \underline{v}_{i,0}$, $N \underline{v}_{i,2} = \underline{v}_{i,1}$, ...

.. $\underline{v}_{i,k} = \underline{u}_i$

In general: $N \underline{v}_{i,j} = \begin{cases} \underline{0} & j = 0 \\ \underline{v}_{i,j-1} & j \neq 0 \end{cases}$

Clear: $\{ \underline{v}_{i,j} \}_{j=0}^{k_i}$ is a Jordan block,

$B = \bigcup_{i \in I} \{\underline{v}_{i,j}\}_{j=0}^{k_i} \leftrightarrow$ direct sum of Jordan blocks

$\mathcal{B}$ is indep by lemma: $N(\mathcal{B}) \subset \mathcal{B} \cup \{\underline{0}\}$
By N-invariant, N does not mix blocks, within each block it's the example above.

Claim: For each k, $\mathrm{Span}_F(\mathcal{B}) \supset \mathrm{Ker}(N^k)$
(for $k=d$, $N^d = 0$, $\mathrm{Span}_F(\mathcal{B}) \supset V$)

Pf: clear for $k=0$. Suppose $\mathrm{Span}_F(\mathcal{B}) \supset \mathrm{Ker}(N^k)$

Let $\underline{v} \in \mathrm{Ker}(N^{k+1})$. Then $N^k \underline{v} \in \mathrm{Ker}(N) \cap \mathrm{Im}(N^k) = W_k$.

so $$N^k \underline{v} = \sum_{i: k_i \geq k} \alpha_i \underline{v}_i \quad \text{for some } \alpha_i.$$
$$= \sum_{i: k_i \geq k} \alpha_i N^k \underline{u}_i$$

$$\Rightarrow N^k \Big(\underline{v} - \sum_{i: k_i \geq k} \alpha_i \underline{u}_i\Big) = 0$$

so $\underline{v} - \sum_{i : k_i \geq k} a_i \underline{u}_i \in \ker(N^k) \subset \mathrm{Span}_F(B)$

by induction hypothesis.

Also $\underline{u}_i \in B$ $\quad \underline{u}_i = \underline{v}_{i,k_i} \in B$

so $\underline{v} \in \mathrm{Span}_F(B)$ and we are done. $\blacksquare$

<u>Defn</u> A <u>Jordan basis</u> is a basis $\{\underline{v}_{i,j}\}_{0 \leq j < k_i}$

s.t. $N\underline{v}_{i,j} = \begin{cases} \underline{v}_{i,j-1} & j \geq 1 \\ \underline{0} & j = 0. \end{cases}$

<u>Lemma:</u> A Jordan basis has exactly

$\dim_F W_{k-1} - \dim_F W_k$

blocks of size k.

$\Leftarrow$ up to permuting blocks, the Jordan form (= matrix of N in basis) is unique.

Pf: In Jordan basis, $\operatorname{Ker} N = \operatorname{Span}\{\underline{v}_{i,0}\}$

$\left\{\underline{v}_{i,j} : \begin{array}{l} j \leq k_i - k \\ k_i \geq k \end{array}\right\}_{CB}$ is a basis for $\operatorname{Im}(N^k)$

$\Rightarrow \operatorname{Im}(N^k) \cap \operatorname{Ker}(N) = \operatorname{Span}_F\{\underline{v}_{i,0} \mid k_i \geq k\}$.

$\blacksquare$

Return to case of general $T \in \operatorname{End}_F(V)$

Theorem: Suppose m_T splits in F.
Then $\exists$ basis $\left\{v_{\lambda,i,j} \;\middle|\; \begin{array}{l} \lambda \in \operatorname{Spec}_F(T) \\ i \in I_\lambda,\ 0 \leq j \leq k(\lambda, i) \end{array}\right\}$

such that

$$(T-\lambda)\underline{v}_{\lambda,i,j} = \begin{cases} v_{\lambda,i,j-1} & j \geq 1 \\ \underline{0} & j = 0 \end{cases}$$

$\Leftrightarrow$

$$T\underline{v}_{\lambda,i,j} = \lambda \underline{v}_{\lambda,i,j} + \begin{cases} \underline{v}_{\lambda,i,j-1} & j \geq 1 \\ \underline{0} & j = 0 \end{cases}$$

Furthermore, if we write $W_\lambda = \operatorname{Ker}(T-\lambda)$,

$\#i$ s.t. $k(\lambda, i) = k$ is

$$\dim_F\left((T-\lambda)^{k-1} V_\lambda \cap W_\lambda\right) - \dim_F\left((T-\lambda)^k V_\lambda \cap W_\lambda\right).$$

Pf: Apply the previous thm to $T|_{V_\lambda} - \lambda$.

Cor: The algebraic multiplicity of λ is $\dim V_\lambda$.

The geometric mult. is # λ-blocks.

Example: $A_1 = I + A$ (A from above)

this has char poly $(x-1)^3$, min poly $(x-1)^2$, back in previous example, with Jordan form

$$\begin{pmatrix} \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} & \\ & (1) \end{pmatrix}$$

If $A_2 = 2I + A$, get Jordan form

$$\begin{pmatrix} \begin{pmatrix} 2 & 1 \\ & 2 \end{pmatrix} & \\ & (2) \end{pmatrix}$$

block-diag matrix, each block has e.v. λ on diag, 1st right above 0 else where.