Math 412, Supplemental lecture

Review: the Jordan Canonical form

Setup: $T \in \mathrm{End}_F(V)$, $\dim_F V = n < \infty$.

Step 1: Find $p_T(x) = \det(x - T)$

$\rightsquigarrow$ Spectrum $\mathrm{Spec}_F(T)$

Extend scalars (if needed) so p_T splits

Step 2: Find generalized eigenspaces

$$V_\lambda = \mathrm{Ker}\left((T-\lambda)^{k_\lambda}\right), \quad k_\lambda \text{ large enough.}$$

Step 3: Inside each V_λ, study $N = (T-\lambda)\restriction_{V_\lambda}$.

Set $W_\lambda = \mathrm{Ker}(T-\lambda)$, $W_\lambda^k = \mathrm{Im}(N^k) \cap W_\lambda$.

$$W_\lambda = W_\lambda^0 \supset W_\lambda^1 \supset \cdots$$

Find basis of W_λ compatible

with filtration

<u>Step 4:</u> For each such basis vector, build Jordan block.

<u>Example 1:</u> $B = \begin{pmatrix} 7 & 1 & 2 & 2 \\ 1 & 4 & -1 & -1 \\ -2 & 1 & 5 & -1 \\ 1 & 1 & 2 & 8 \end{pmatrix}$.

Check: $p_B(x) = (x-6)^4$, let $N = B - 6I$

$$= \begin{pmatrix} 1 & 1 & 2 & 2 \\ 1 & -2 & -1 & -1 \\ -2 & 1 & -1 & -1 \\ 1 & 1 & 2 & 2 \end{pmatrix}$$

$$N^2 = \begin{pmatrix} 0 & 3 & 3 & 3 \\ 0 & 3 & 3 & 3 \\ 0 & -6 & -6 & -6 \\ 0 & 3 & 3 & 3 \end{pmatrix}, \quad N^3 = 0$$

Gaussian elimination: $N = E \begin{pmatrix} 3 & -3 & 0 & 0 \\ 1 & -2 & 1 & 1 \\ -3 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$$\Rightarrow \ker N = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \;\middle|\; \begin{matrix} x - y = 0 \\ x - 2y + z + w = 0 \end{matrix} \right\}$$

$$= \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \;\middle|\; x = y = -(z+w) \right\}$$

$$\operatorname{Im}(N^2) = \operatorname{Span}\left\{\begin{pmatrix}1\\1\\-2\\1\end{pmatrix}\right\}$$

E.g. $$N^2\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = N\begin{pmatrix}2\\-1\\-1\\2\end{pmatrix} = \begin{pmatrix}3\\3\\-6\\3\end{pmatrix}.$$

So ~~set~~ block $\left\{\begin{pmatrix}3\\3\\-6\\3\end{pmatrix}, \begin{pmatrix}2\\-1\\-1\\2\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right\}$

Also $\begin{pmatrix}-1\\-1\\1\\0\end{pmatrix} \in \operatorname{Ker} N$, indep $\begin{pmatrix}3\\3\\-6\\3\end{pmatrix}$.

So Jordan basis is $\left\{\begin{pmatrix}3\\3\\-6\\3\end{pmatrix}, \begin{pmatrix}2\\-1\\-1\\2\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \begin{pmatrix}-1\\-1\\1\\0\end{pmatrix}\right\}$

Matrix of N $\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$

So Jordan form of B is $\begin{pmatrix}\begin{pmatrix}6&1&0\\0&6&1\\0&0&6\end{pmatrix} & 0\\ 0 & (6)\end{pmatrix}.$

Example 2: $C = \begin{pmatrix}4&0&1&0\\2&2&3&0\\-1&0&2&0\\4&0&1&2\end{pmatrix}$. $p_C(x) = (x-2)^2(x-3)^2$

$$C-2I = \begin{pmatrix}2&0&1&0\\2&0&3&0\\-1&0&0&0\\4&0&1&0\end{pmatrix}$$

$$\operatorname{Ker}(C-2I) = \left\{\begin{pmatrix}x\\y\\z\\w\end{pmatrix} : \begin{matrix}2x+z=0\\2x+3z=0\\4x+z=0\end{matrix}\right\}$$

$$= \left\{\begin{pmatrix}0\\y\\0\\w\end{pmatrix}\right\}$$

Eigenvectors $\begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}.$

$$C-3I = \begin{pmatrix}1&0&1&0\\2&-1&3&0\\-1&0&-1&0\\4&0&1&-1\end{pmatrix}, \quad (C-3I)^2 = \begin{pmatrix}0&0&0&0\\-3&1&4&0\\0&0&0&0\\-1&0&2&1\end{pmatrix}$$

$$\ker(C-3I) = \operatorname{Span}\left\{\begin{pmatrix}1\\-1\\-1\\3\end{pmatrix}\right\}$$

$$\ker(C-3I)^2 = \left\{\begin{pmatrix}x\\y\\z\\w\end{pmatrix} \middle| \begin{array}{l} y = 3x-4z \\ w = x-z \end{array}\right\}$$

$$= \operatorname{Span}\left\{\begin{pmatrix}1\\-1\\-1\\3\end{pmatrix}, \begin{pmatrix}1\\3\\0\\1\end{pmatrix}\right\}$$

$$(C-3I)\begin{pmatrix}1\\3\\0\\1\end{pmatrix} = \begin{pmatrix}1\\-1\\-1\\3\end{pmatrix}.$$

$$\Rightarrow \left\{\begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\-1\\3\end{pmatrix}, \begin{pmatrix}1\\3\\0\\1\end{pmatrix}\right\}$$

Jordan form $\begin{pmatrix}(2)&&0\\&(2)&\\0&&\begin{pmatrix}3&1\\&3\end{pmatrix}\end{pmatrix}.$