Math 412, Lecture 17

Part 3: Analysis on Vector spaces

For the rest of the course, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

Norms on Vector spaces

Metrics

Def: A metric space is a pair (X, d_X) where $d_X : X \times X \to \mathbb{R}_{\geq 0}$ s.t.

(1) $d(x,y) = d(y,x)$

(2) $d(x,y) = 0$ iff $x = y$

(3) (triangle inequality) $d(x,z) \leq d(x,y) + d(y,z)$

(pseudometric if weaken (2) to $d(x,x) = 0$)

Notation: The ball of radius r is

$$B_X(x,r) = \{ y \in X \mid d(x,y) \leq r \}$$

($B^o_X(x,r) = \{ y \mid d(x,y) < r \}$ is the open ball)

Def: let (X, d_X) (Y, d_Y) be metric spaces. Say $f: X \to Y$ is

(1) continuous if $\forall x \, \forall \varepsilon > 0 \, \exists \delta > 0 : \forall y$
$d_X(x,y) < \delta \to d_Y(f(x), f(y)) < \varepsilon.$

(2) uniformly cts if $\forall \varepsilon_0 \, \exists \delta > 0 \, \forall x \forall y$ if $d_X(x,y) < \delta$ then $d_Y(f(x), f(y)) < \varepsilon$

(3) Lipschitz cts if $\exists L > 0 \, \forall x \forall y : d_Y(f(x), f(y)) \leq$
$\leq L \cdot d_X(x,y)$

Clearly $(3) \Rightarrow (2) \Rightarrow (1)$.

Set: $\|f\|_{Lip} = \underset{L}{\text{smallest}} = \sup\limits_{x \neq y} \dfrac{d_Y(f(x), f(y))}{d_X(x,y)}$

Lemma: If f, g composable, both of type (1), (2), or (3), then $f \circ g$ of same type.

$$\|f \circ g\|_{Lip} \leq \|f\|_{Lip} \|g\|_{Lip}.$$

Def: Call (X, d) complete if every Cauchy sequence in X has a limit.

Vector spaces: Norms

Fix vsp V.

Def: A norm on V is a map $\|\cdot\| : V \to \mathbb{R}_{\geq 0}$ s.t.

(1) $\|\underline{v}\| = 0$ iff $\underline{v} = \underline{0}$

(2) $\|\alpha \underline{v}\| = |\alpha| \cdot \|\underline{v}\|$

(3) $\|\underline{u} + \underline{v}\| \leq \|\underline{u}\| + \|\underline{v}\|$.

Lemma: $d(\underline{u}, \underline{v}) = \|\underline{v} - \underline{u}\|$ is a metric on V

Conversely, if d is a metric on V which is translation-invariant $(d(\underline{x}, \underline{y}) = d(\underline{x} + \underline{u}, \underline{y} + \underline{u}))$ and 1-homogeneous $d(\alpha \underline{x}, \alpha \underline{y}) = |\alpha| \, d(\underline{x}, \underline{y}))$ then d comes from a norm.

Clear: If $W \subset V$ is a subspace, $\|\cdot\|_{W}$ is a norm on W.

Examples: standard norms on $\mathbb{F}^n$

(1) supremum norm $\|v\|_\infty = \max\{|v_i|\}_{i=1}^n$.

↳ uniform convergence.

(2) Manhattan norm $\|\underline{v}\|_1 = \sum_{i=1}^n |v_i|$

(3) Euclidean norm $\|\underline{v}\|_2 = \left(\sum_{i=1}^n |v_i|^2\right)^{1/2}$

(comes from inner pdt $\langle \underline{u}, \underline{v} \rangle = \sum_{i=1}^n \bar{u}_i v_i$.)

(4) ℓ^p : $\|\underline{v}\|_p = \left(\sum_{i=1}^n |v_i|^p\right)^{1/p}$, $1 \le p < \infty$.

(Ex: $\lim_{p\to\infty} \|\underline{v}\|_p = \|\underline{v}\|_\infty$)

Geometrically, can relate norm to its unit ball

$\{\underline{v} \mid \|\underline{v}\| \le 1\}$.

Examples: In infinite dimensions, norm comes first, defines the vsp

Def: $\ell^\infty(X) = \{f : X \to \mathbb{F} \mid \sup\{|f(x)| : x \in X\} < \infty\}$

equipped with norm $\|f\|_\infty = \sup_{x \in X} |f(x)|$

Better: Define $\|f\|_\infty$ for all f, maybe with value ∞, check: $\{f \in F^X \mid \|f\| < \infty\}$ is a subspace.

Remark: A vsp V with basis B, can be embedded in $\ell^\infty(B)$ via coeff.

Example: $\ell^p(\mathbb{N}) = \{ \underline{a} \in F^{\mathbb{N}} : \sum_{i=1}^{\infty} |a_i|^p < \infty \}$

$$\|\underline{a}\|_p = \left(\sum_{i=1}^{\infty} |a_i|^p \right)^{1/p}.$$

check: $\ell^p(\mathbb{N}) \subseteq \ell^q(\mathbb{N})$ iff $1 \leq p \leq q \leq \infty$.

Example: $L^p(\mathbb{R}) = \{ f : \mathbb{R} \to F \text{ measurable} \mid \int_{\mathbb{R}} |f(x)|^p dx < \infty \} \Big/ \{ f \mid \underset{a.e.}{f = 0} \}$

Theorem: (Elliptic regularity) Let $\Omega \subset \mathbb{R}^2$ be open, let $f \in L^2(\Omega)$ satisfy $\Delta f = \lambda f$ distributionally:

$$\forall g \in C_c^\infty(\Omega), \quad \int_\Omega f \cdot \Delta g = \lambda \int_\Omega fg$$

Then there is a smooth function $\tilde{f}$ s.t. $f = \tilde{f}$ a.e.

Then $\Delta \tilde{f} = \lambda \tilde{f}$ pointwise.

$\blacksquare$

Def: Say $\underline{v}_n \to \underline{v}$ if $\|\underline{v}_n - v\| \xrightarrow[n\to\infty]{} 0$

Fact: On $\mathbb{R}^n$ all norms define same notion of convergence.

Lemma: All norms on $\mathbb{R}^n$ are cts (wrt $\|\cdot\|_1$)

Pf:

$$\|\underline{x}\| = \left\| \sum_{i=1}^n x_i \underline{e}_i \right\| \le \sum_{i=1}^n |x_i| \cdot \|\underline{e}_i\| \le M \cdot \|\underline{x}\|_1$$

std basis of $\mathbb{R}^n$

if $M = \max_i \|\underline{e}_i\|$

so $\big| \|\underline{x}\| - \|\underline{y}\| \big| \le \|\underline{x} - \underline{y}\| \le M \|\underline{x} - \underline{y}\|_1$

$\blacksquare$

Def: Call $\|\cdot\|$, $\|\cdot\|'$ equivalent if $\exists\, 0<m<M$ s.t.

$$m\|\underline{x}\| \le \|\underline{x}\|' \le M\|\underline{x}\|$$

Ex: This is an equivalence relation. Two norms equiv iff agree on set of sequences s.t. $\underline{x}_n \to \underline{0}$.

Theorem: All norms on $\mathbb{R}^n$ are equivalent.

Pf: let $\|\cdot\|$ be any norm. Then $\|\underline{x}\| \le M\cdot\|\underline{x}\|_1$.

let $S = \{\underline{x} : \|\underline{x}\|_1 = 1\}$ ("ℓ^1 sphere")

S is closed and bounded so compact.

$\Rightarrow$ $m = \min\{\|\underline{x}\| : \underline{x}\in S\}$ exists

$m>0$: $\underline{0}\notin S$ so no $\underline{x}\in S$ has $\|\underline{x}\| = 0$.

now if $\underline{x}\in\mathbb{R}^n$, non zero, $\left\|\dfrac{\underline{x}}{\|\underline{x}\|_1}\right\|_1 = 1$

$\Rightarrow \left\|\dfrac{\underline{x}}{\|\underline{x}\|_1}\right\| \ge m \Rightarrow \|\underline{x}\| \ge \|\underline{x}\|_1\, m.$

$\Rightarrow \quad m\|\underline{x}\|_1 \leq \|\underline{x}\| \leq M\|\underline{x}\|_1$

$\blacksquare$

Norm on linear maps: operator norm

<u>Def</u>: Let U, V be normed vector spaces. A linear map $T \in \mathrm{Hom}_F(U,V)$ is **bounded** if $\exists M \geq 0$ s.t.

$$\forall \underline{u} \in U: \quad \|T\underline{u}\|_V \leq M \cdot \|\underline{u}\|_U.$$

The **operator norm** $\|T\| = \|T\|_{U\to V}$ is the least such M

<u>Example</u>: Say U is the space of initial conditions for some evolution. Let V be the space of possible states at time t. Say $T: U \to V$ is time evolution

Then T being <u>bounded</u> reflects <u>stability</u> of PDE.

<u>Example</u>: $\mathrm{Id}: U \to U$ has $\|\mathrm{Id}\|_{U\to U} = 1$

Example: $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ acting on $\mathbb{R}^2$:

$$\left\| A \begin{pmatrix} x \\ y \end{pmatrix} \right\|_1 = \left\| \begin{pmatrix} x+y \\ y \end{pmatrix} \right\|_1 = |x+y| + |y| \leq 2(|x|+|y|)$$
$$= 2 \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|_1$$

so $\|A\|_1 = 2$. (equality if $x=0$)

$$\left\| A \begin{pmatrix} x \\ y \end{pmatrix} \right\|_2^2 = \left\| \begin{pmatrix} x+y \\ y \end{pmatrix} \right\|_2^2 = (x+y)^2 + y^2 = x^2 + 2xy + 2y^2$$
$$\leq \frac{3+\sqrt{5}}{2}(x^2+y^2)$$

$$\Rightarrow \|A\|_2 = \sqrt{\frac{3+\sqrt{5}}{2}}$$

equality if $x = y = 1$

$$\left\| A \begin{pmatrix} x \\ y \end{pmatrix} \right\|_\infty = \max\{|x+y|, |y|\} \leq 2 \max\{|x|, |y|\}$$
$$= 2 \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|_\infty$$

so $\|A\|_\infty = 2$

(can also try $\|A\|_{\ell^1 \to \ell^3}$, $\|A\|_{\ell^\infty \to \ell^2}$, $-$)

Example: $D_x : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$
never bounded:

$D_x\left(e^{\lambda x}\right) = \lambda e^{\lambda x}$

so $\|D_x\| \geq |\lambda|$ for any norm on $C^\infty(\mathbb{R})$

so D_x is unbounded

Lemma: Any map on f.d. space is bounded.

Pf: Identify U with $\mathbb{R}^n$, then $\|\cdot\|_u$ is equivalent to $\|\cdot\|_1$, so $\|\underline{u}\|_1 \leq A \cdot \|\underline{u}\|_u$.

Now $\|T\underline{u}\|_V = \|T(\sum_{i=1}^n u_i \underline{e}_i)\|_V$

$$\leq \sum_{i=1}^n |u_i| \, \|T\underline{e}_i\|_V \qquad B = \max_i \|T\underline{e}_i\|_V$$

$$\leq B \sum_{i=1}^n |u_i| \leq B \cdot \|\underline{u}\|_1 \leq BA \|\underline{u}\|_u$$

$\Rightarrow \|T\underline{u}\|_V \leq (BA) \|\underline{u}\|_u$

$\blacksquare$

Lemma: Let S, T be composably bounded. Then ST is bounded, $\|ST\| \leq \|S\| \cdot \|T\|$.

Pf: Say $T: U \to V$, $S: V \to W$.

$\forall \underline{u}$: $\|(ST)\underline{u}\|_W = \|S(T\underline{u})\|_W$

$\leq \|S\|_{V\to W} \cdot \|T\underline{u}\|_V$

$\leq \|S\|_{V\to W} \|T\|_{U\to V} \cdot \|\underline{u}\|_U$

∎

Prop: The operator norm is a norm on

$\mathrm{Hom}_b(U, V) = \{$bounded linear maps$\}$.

Pf: Let $S, T \in \mathrm{Hom}_b(U, V)$. Then

$\|(\alpha S + T)\underline{u}\|_V = \|\alpha S\underline{v} + T\underline{u}\|_V$

$\leq |\alpha| \|S\underline{u}\|_V + \|T\underline{u}\|_V \leq (|\alpha| \cdot \|S\| + \|T\|) \|\underline{u}\|_U$

$\Rightarrow \alpha S + T$ is bdd, and $\|\alpha S + T\| \leq |\alpha|\, \|S\| + \|T\|$

so $\|S + T\| \leq \|S\| + \|T\|$, $\|\alpha S\| \leq |\alpha| \cdot \|S\|$

Also get $\|S\| = \|\frac{1}{\alpha} \cdot \alpha S\| \leq \frac{1}{|\alpha|} \|\alpha S\|$

$\Rightarrow \|\alpha S\| \geq |\alpha| \cdot \|S\|$ so $\|\alpha S\| = |\alpha| \cdot \|S\|$

$(0 \in \mathrm{Hom}_b(U, V))$ and if $T \neq 0$, $\exists \underline{u} : T\underline{u} \neq \underline{0}$

then $\|T\underline{u}\| \neq 0$ so $\frac{\|T\underline{u}\|_V}{\|\underline{u}\|_U} > 0$

so $\|T\| \geq \frac{\|T\underline{u}\|_V}{\|\underline{u}\|_U} > 0.$

$\blacksquare$