

Lior Silberman's Math 412: Problem Set 1 (due 14/1/2025)

Practice problems numbered M1 etc), any sub-parts marked “PRAC” (practice) “SUPP” (supplementary) and supplementary problems are not for submission.

Practice problems

- M1. Show that the map $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ given by $f(x, y, z) = x - 2y + z$ is a linear map. Show that the maps $(x, y, z) \mapsto 1$ and $(x, y, z) \mapsto x^2$ are not.
- M2. Let F be a field, X a set. Carefully show that pointwise addition and scalar multiplication endow the set F^X of functions from X to F with the structure of an F -vectorspace. Extend your arguments to V^X where V is an F -vectorspace.
- M3. For $x \in X$ let $\delta_x: F^X \rightarrow F$ be evaluation at x : $\delta_x(f) \stackrel{\text{def}}{=} f(x)$. Show that each δ_x is a linear map. Meditate on the fact that the vector space structure was defined exactly so that δ_x are linear maps.

For submission

1. (Vector space axioms) Let $E = (E, 0_E, 1_E, +_E, \cdot_E)$ be a field, and let $F \subset E$ be a subfield, in other words a subset which is closed under the operations and satisfies the field axioms. It is commonly said that E naturally has the structure of an F -vectorspace.
- (a) Explicitly write the quadruple that was implicitly introduced in the statement of the problem.
- (b) Verify the vector space axioms for this putative vector space structure.
(Hint: this problem is tedious rather than hard)

RMK The following problem introduces a device for showing that sets of vectors are linearly independent. Make sure you understand how this argument works.

2. Let V be a vector space, $S \subset V$ a set of vectors. A *minimal dependence* in S is an equality $\sum_{i=1}^m a_i v_i = \underline{0}$ where $v_i \in S$ are distinct, a_i are scalars not all of which are zero, and $m \geq 1$ is as small as possible so that such $\{a_i\}, \{v_i\}$ exist.
- It is implicit in the following that either S is independent or it has a minimal dependence. Make this explicit in your mind (don't write this bit up).

PRAC Find a minimal dependence among $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \right\} \subset \mathbb{R}^3$.

- (a) Show that in a minimal dependence the a_i are all non-zero.
- (b) Suppose that $\sum_{i=1}^m a_i v_i$ and $\sum_{i=1}^m b_i v_i$ are minimal dependences in S , involving the exact same set of vectors. Show that there is a non-zero scalar c such that $a_i = c b_i$.
- (c) Let $T: V \rightarrow V$ be a linear map, and let $S \subset V$ be a set of (non-zero) eigenvectors of T , each corresponding to a distinct eigenvalue. Applying T to a minimal dependence in S obtain a contradiction to (c) and conclude that S is actually linearly independent.
- (*d) Let Γ be a group. The set $\text{Hom}(\Gamma, \mathbb{C}^\times)$ of group homomorphisms from Γ to the multiplicative group of nonzero complex numbers is called the set of *quasicharacters* of Γ (the notion of “character of a group” has an additional, different but related meaning, which is not at issue in this problem). Show that $\text{Hom}(\Gamma, \mathbb{C}^\times)$ is linearly independent in the space \mathbb{C}^Γ of functions from Γ to \mathbb{C} .

RMK For every prime power $q = p^r$ there is a field \mathbb{F}_q with q elements, and two such fields are isomorphic. They are usually called *finite fields*, but also *Galois fields* after their discoverer.

Supplementary Problems I: Fields and subfields

- A. Let $\mathbb{Q}(\sqrt{2})$ denote the set $\{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\} \subset \mathbb{R}$.
- (a) Show that $\mathbb{Q}(\sqrt{2})$ is a \mathbb{Q} -subspace of \mathbb{R} .
 - (b) Show that $\mathbb{Q}(\sqrt{2})$ is two-dimensional as a \mathbb{Q} -vector space. In fact, identify a basis.
 - (*c) Show that $\mathbb{Q}(\sqrt{2})$ is a field.
 - (**d) Let V be a vector space over $\mathbb{Q}(\sqrt{2})$ and suppose that $\dim_{\mathbb{Q}(\sqrt{2})} V = d$. Show that $\dim_{\mathbb{Q}} V = 2d$.
- B. Let V be an abelian group. The *Minkowski sum* of $A, B \subset V$ is the subset $A + B \stackrel{\text{def}}{=} \{\underline{a} + \underline{b} \mid \underline{a} \in A, \underline{b} \in B\}$.
- (a) Let $A, B \subset \mathbb{R}$ be finite sets. Show that there exists $c \in A + B$ which has a unique representation as $a + b$ with $a \in A, b \in B$.
 - (b) The claim of part (a) is also true in the free abelian group \mathbb{Z}^d . Prove this by constructing a \mathbb{Q} -linear embedding $\mathbb{Q}^d \rightarrow \mathbb{R}$.

Supplementary Problems II: How physicists define vectors

Fix a field F .

- C. (The general linear group)
- Let $\text{GL}_n(F)$ denote the set of invertible $n \times n$ matrices with coefficients in F . Show that $\text{GL}_n(F)$ forms a group with the operation of matrix multiplication.
 - For a vector space V over F let $\text{GL}(V)$ denote the set of invertible linear maps from V to itself. Show that $\text{GL}(V)$ forms a group with the operation of composition.
 - Suppose that $\dim_F V = n$. Show that $\text{GL}_n(F) \simeq \text{GL}(V)$ (hint: show that each of the two groups is isomorphic to $\text{GL}(F^n)$).
- D. (Group actions) Let G be a group, X a set. An *action* of G on X is a map $\cdot : G \times X \rightarrow X$ such that $g \cdot (h \cdot x) = (gh) \cdot x$ and $1_G \cdot x = x$ for all $g, h \in G$ and $x \in X$ (1_G is the identity element of G).
- Show that matrix-vector multiplication $(g, \underline{v}) \mapsto g\underline{v}$ defines an action of $G = \text{GL}_n(F)$ on $X = F^n$.
 - Let V be an n -dimensional vector space over F , and let \mathcal{B} be the set of ordered bases of V . For $g \in \text{GL}_n(F)$ and $B = \{\underline{v}_j\}_{j=1}^{\dim V} \in \mathcal{B}$ set $gB = \left\{ \sum_{j=1}^n g_{ij} \underline{v}_j \right\}_{i=1}^n$. Check that $gB \in \mathcal{B}$ and that $(g, B) \mapsto gB$ is an action of $\text{GL}_n(F)$ on \mathcal{B} .
 - Show that the action is *transitive*: for any $B, B' \in \mathcal{B}$ there is $g \in \text{GL}_n(F)$ such that $gB = B'$.
 - Show that the action is *simply transitive*: that the g from part (b) is unique.
- E. (From the physics department) Let V be an n -dimensional vector space, and let \mathcal{B} be its set of bases. Given $\underline{u} \in V$ define a map $\phi_{\underline{u}} : \mathcal{B} \rightarrow F^n$ by setting $\phi_{\underline{u}}(B) = \underline{a}$ if $B = \{\underline{v}_i\}_{i=1}^n$ and $\underline{u} = \sum_{i=1}^n a_i \underline{v}_i$.
- Show that $\alpha \phi_{\underline{u}} + \phi_{\underline{u}'} = \phi_{\alpha \underline{u} + \underline{u}'}$. Conclude that the set $\{\phi_{\underline{u}}\}_{\underline{u} \in V}$ forms a vector space over F .
 - Show that the map $\phi_{\underline{u}} : \mathcal{B} \rightarrow F^n$ is *equivariant* for the actions of B(a), B(b), in that for each $g \in \text{GL}_n(F)$, $B \in \mathcal{B}$, $g(\phi_{\underline{u}}(B)) = \phi_{\underline{u}}(gB)$.
 - Physicists define a “covariant vector” to be an equivariant map $\phi : \mathcal{B} \rightarrow F^n$. Let Φ be the set of covariant vectors. Show that the map $\underline{u} \mapsto \phi_{\underline{u}}$ defines an isomorphism $V \rightarrow \Phi$. (Hint: define a map $\Phi \rightarrow V$ by fixing a basis $B = \{\underline{v}_i\}_{i=1}^n$ and mapping $\phi \mapsto \sum_{i=1}^n a_i \underline{v}_i$ if $\phi(B) = \underline{a}$).
 - Physicists define a “contravariant vector” to be a map $\phi : \mathcal{B} \rightarrow F^n$ such that $\phi(gB) = {}^t g^{-1} \cdot (\phi(B))$. Verify that $(g, \underline{a}) \mapsto {}^t g^{-1} \underline{a}$ defines an action of $\text{GL}_n(F)$ on F^n , that the set Φ' of contravariant vectors is a vector space, and that it is naturally isomorphic to the dual vector space V' of V .

Supplementary Problems III: Fun in positive characteristic

- F. Let F be a field of characteristic 2 (that is, $1_F + 1_F = 0_F$).
- Show that for all $x, y \in F$ we have $x + x = 0_F$ and $(x + y)^2 = x^2 + y^2$.
 - Considering F as a vector space over \mathbb{F}_2 as in problem 5, show that the field automorphism $\text{Frob} : F \rightarrow F$ given by $\text{Frob}(x) = x^2$ is an \mathbb{F}_2 -linear map.
 - Suppose that the map $x \mapsto x^2$ is actually F -linear and not only \mathbb{F}_2 -linear. Show that $F = \mathbb{F}_2$. RMK Compare your answer with practice problem 1.
- G. (This problem requires a bit of number theory) Now let F have characteristic $p > 0$. Show that the *Frobenius endomorphism* $x \mapsto x^p$ is \mathbb{F}_p -linear.