Lior Silberman's Math 412: Problem Set 2 (due 23/1/2025)

Practice

M1 Let $\{V_i\}_{i\in I}$ be a family of vector spaces, and let $A_i \in \mathrm{End}(V_i) = \mathrm{Hom}(V_i, V_i)$.

(a) Show that there is a unique element $\bigoplus_{i\in I} A_i \in \mathrm{End}(\bigoplus_{i\in I} V_i)$ whose restriction to the image of V_i in the sum is A_i.

(b) Carefully show that the matrix of $\bigoplus_{i\in I} A_i$ in an appropriate basis is block-diagonal.

Direct sums

1. (Counterexamples)

(a) Construct a vector space W and three subspaces $U, V_1, V_2 \subset W$ such that $W = U \oplus V_1 = U \oplus V_2$ (internal direct sums) but $V_1 \neq V_2$.

(b) Give an example of $V_1, V_2, V_3 \subset W$ where $V_i \cap V_j = \{\underline{0}\}$ for every $i \neq j$ yet the sum $V_1 + V_2 + V_3$ is not direct.

2. (Diagonability)

DEF A square matrix $A \in M_n(F)$ is diagonable (over F) if there exists an invertible matrix $S \in \mathrm{GL}_n(F)$ such that SAS^{-1} is diagonal.

(a) Show that $A \in M_n(F)$ is diagonable iff there exist n one-dimensional subspaces $V_i \subset F^n$ such $F^n = \bigoplus_{i=1}^n V_i$ and $A(V_i) \subset V_i$ for all i.

(b) Let $T \in \mathrm{End}_F(V)$. For each $\lambda \in F$ let $V_\lambda = \mathrm{Ker}(T - \lambda)$ be the corresponding eigenspace. Let $\mathrm{Spec}_F(T) = \{\lambda \in F \mid V_\lambda \neq \{0\}\}$ be the set of eigenvalues of T. Show that the sum $\sum_{\lambda\in\mathrm{Spec}_F(T)} V_\lambda$ is direct.

(c) Call $T \in \mathrm{End}_F(V)$ *diagonable* if its matrix with respect to some basis is diagonable. Show that T is diagonable iff $\sum_{\lambda\in\mathrm{Spec}_F(T)} V_\lambda = V$.

Direct products

CONSTRUCTION. *Let $\{V_i\}_{i\in I}$ be a (possibly infinite) family of vector spaces.*

(1) *The* external direct product $\prod_{i\in I} V_i$ *is the vector space whose underlying space is* $\{f\colon I \to \bigcup_{i\in I} V_i \mid \forall i : f(i) \in V_i\}$ *with the operations of pointwise addition and scalar multiplication.*

(2) *The* external direct sum $\bigoplus_{i\in i} V_i$ *is the subspace of finitely supported functions* $\left\{f \in \prod_{i\in I} V_i \mid \#\left\{i \mid f(i) \neq \underline{0}_{V_i}\right\} < \infty\right\}$.

3. (Tedium)

(a) Show that the direct product is a vector space

(b) Show that the direct sum is a subspace.

(c) Let $\pi_i\colon \prod_{j\in I} V_j \to V_i$ be the projection on the ith coordinate ($\pi_i(f) = f(i)$). Show that π_i are surjective linear maps.

(d) Let $\sigma_i\colon V_i \to \prod_{i\in I} V_i$ be the map such that $\sigma_i(\underline{v})(j) = \begin{cases} \underline{v} & j = i \\ \underline{0} & j \neq i \end{cases}$.

Show that σ_i are injective linear maps.

SUPP (Direct sums) Show that $\bigoplus_{i\in I} V_i$ is the internal direct sum of the images $\sigma_i(V_i)$ and conclude that direct sums of vector spaces exist.

4. (Meat)
   (a) Let Z be another vector space, and suppose we have for each i a linear map $g_i \in \mathrm{Hom}(Z, V_i)$. Show that there is a unique $g \in \mathrm{Hom}(Z, \prod_i V_i)$ such that $\pi_i \circ g = g_i$ for all i.
   DEF A vector space P equipped with maps $\pi_i' \colon P \to V_i$ with the property of part (a) is called a *direct product* of the V_i.
   RMK In this language part (a) shows that direct products exist.
   (b) Show that any two direct products are uniquely isomorphic compatibly with the projection maps.
   (c) Show that if P is a direct product then the maps π_i' are surjective.

Quotients

5. Write $M_n(F)$ for the space of $n \times n$ matrices with entries in F. Let $\mathfrak{sl}_n(F) = \{A \in M_n(F) \mid \mathrm{Tr} A = 0\}$ and let $\mathfrak{pgl}_n(F) = M_n(F)/F \cdot I_n$ (matrices modulu scalar matrices). Suppose that n is invertible in F (equivalently, that the characteristic of F does not divide n). Show that the quotient map $M_n(F) \to \mathfrak{pgl}_n(F)$ restricts to an isomorphism $\mathfrak{sl}_n(F) \to \mathfrak{pgl}_n(F)$.

6. For $f \colon \mathbb{R}^n \to \mathbb{R}$ the *Lipschitz constant* of f is the (possibly infinite) number
$$\|f\|_{\mathrm{Lip}} \stackrel{\mathrm{def}}{=} \sup\left\{ \frac{|f(x) - f(y)|}{|x-y|} \mid x, y \in \mathbb{R}^n, x \neq y \right\}.$$
   Let $\mathrm{Lip}(\mathbb{R}^n) = \left\{ f \colon \mathbb{R}^n \to \mathbb{R} \mid \|f\|_{\mathrm{Lip}} < \infty \right\}$ be the space of *Lipschitz functions*.
   PRA Show that $f \in \mathrm{Lip}(\mathbb{R}^n)$ iff there is C such that $|f(x) - f(y)| \leq C\,|x-y|$ for all $x, y \in \mathbb{R}^n$.
   (a) Show that $\mathrm{Lip}(\mathbb{R}^n)$ is a subspace of the space of functions on $\mathbb{R}^n$.
   (b) Let $\mathbb{1}$ be the constant function 1. Show that $\|f\|_{\mathrm{Lip}}$ descends to a function on $\mathrm{Lip}(\mathbb{R}^n)/\mathbb{R}\mathbb{1}$.
   (c) For $\bar{f} \in \mathrm{Lip}(\mathbb{R}^n)/\mathbb{R}\mathbb{1}$ show that $\left\|\bar{f}\right\|_{\mathrm{Lip}} = 0$ iff $\bar{f} = 0$.

Supplement: Quotients and complements

A. (Quotients and complements) Let W be a vector space and let $U \subset W$ be a subspace.
   (a) Show that there exists another subspace $V \subset W$ such that $W = U \oplus V$.
   DEF We say V is a *complement* for U (in W).
   (b) Let V be a complement for U and let $\pi \colon W \to W/U$ be the quotient map. Show that the restriction of π to V is an isomorphism.
   (c) Conclude that if V_1, V_2 are both complements then $V_1 \simeq V_2$ (c.f. problem P2)
   REM A subspace will have many complements, while the quotient is "canonical".

B. (Structure of quotients) Let $V \subset W$ with quotient map $\pi \colon W \to W/V$.
   (a) Show that mapping $U \mapsto \pi(U)$ gives a bijection between (1) the set of subspaces of W containing V and (2) the set of subspaces of W/V.
   (b) (The universal property) Let Z be another vector space. Show that $f \mapsto f \circ \pi$ gives a linear bijection $\mathrm{Hom}(W/V, Z) \to \{g \in \mathrm{Hom}(W, Z) \mid V \subset \mathrm{Ker}\, g\}$.

Supplement: more universal properties

C. A *free abelian group* is a pair (F,S) where F is an abelian group, $S \subset F$, and ("universal property") for any abelian group A and any (set) map $f\colon S \to A$ there is a unique group homomorphism $\bar{f}\colon G \to A$ such that $\bar{f}(s) = f(s)$ for any $s \in S$. The size $\#S$ is called the *rank* of the free abelian group.
   (a) Show that $(\mathbb{Z}, \{1\})$ is a free abelian group.
   (b) Show that $\left(\mathbb{Z}^d, \{\underline{e}_k\}_{k=1}^{d}\right)$ is a free abelian group.
   (c) Let $(F,S), (F',S')$ be free abelian groups and let $f\colon S \to S'$ be a bijection. Show that f extends to a unique isomorphism $\bar{f}\colon F \to F'$.
   (d) Let (F,S) be a free abelian group. Show that S generates F.
   (e) Show that every element of a free abelian group has infinite order.

D. Let $\{G_i\}_{i\in I}$ be groups. Show that the Cartesian product $\prod_i G_i$ with coordinate-wise operations and with the natural projections $\pi_j\colon \Pi_i G_i \to G_j$ is a direct product, in the sense that it has the universal property of problem 4 (with "vector spaces" replaced by "groups" and "linear maps" by "group homomorphisms".

RMK The "direct sum" object for groups is much more complicated. It is called the "free product".

Supplement: Lipschitz functions

DEFINITION. Let $(X,d_X), (Y,d_Y)$ be metric spaces, and let $f\colon X \to Y$ be a function. We say f is a *Lipschitz function* (or is "Lipschitz continuous") if for some C and for all $x,x' \in X$ we have

$$d_Y\left(f(x), f(x')\right) \le C d_X\left(x,x'\right).$$

E. Write $\mathrm{Lip}(X,Y)$ for the space of Lipschitz continuous functions; for $f \in \mathrm{Lip}(X,Y)$ write $\|f\|_{\mathrm{Lip}} = \sup\left\{\frac{d_Y(f(x),f(x'))}{d_X(x,x')} \mid x \neq x' \in X\right\}$ for its *Lipschitz constant*.
   (a) Show that Lipschitz functions are, indeed, continuous (in fact uniformly continuous).
   (b) Suppose Z is another metric space and that $g\colon Y \to Z$ is also Lipschitz. Show that $g \circ f$ is Lipschitz and that $\|g\circ f\|_{\mathrm{Lip}} \le \|g\|_{\mathrm{Lip}}\|f\|_{\mathrm{Lip}}$.
   (c) Let $f \in C^1(\mathbb{R}^n;\mathbb{R})$. Show that $\|f\|_{\mathrm{Lip}} = \sup\{|\nabla f(x)| : x \in \mathbb{R}^n\}$.
   (d) Generalize problem 6 to the case of $\mathrm{Lip}(X,\mathbb{R})$ where X is any metric space.
   (e) Show that $\mathrm{Lip}(X,\mathbb{R})/\mathbb{R}\mathbb{1}$ is a complete normed space for all metric spaces X.