Lior Silberman's Math 412: Problem Set 3 (due 3/2/2025)

Practice

M1 Let $\underline{u}_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}$, $\underline{u}_2 = \begin{pmatrix}1\\1\\-1\end{pmatrix}$, $\underline{u}_3 = \begin{pmatrix}1\\-1\\-1\end{pmatrix}$, $\underline{u} = \begin{pmatrix}0\\1\\0\end{pmatrix}$ as vectors in $\mathbb{R}^3$.

(a) Construct an explicit linear functional $\varphi \in \left(\mathbb{R}^3\right)'$ vanishing on $\underline{u}_1, \underline{u}_2$.
(b) Show that $\{\underline{u}_1, \underline{u}_2, \underline{u}_3\}$ is a basis on $\mathbb{R}^3$ and find its dual basis.
(c) Evaluate the elements of dual basis at $\underline{u}$.

M2 Let V be n-dimensional and let $\{\varphi_i\}_{i=1}^m \in V'$.

(a) Show that if $m < n$ there is a non-zero $\underline{v} \in V$ such that $\varphi_i(\underline{v}) = 0$ for all i. Interpret this as a statement about linear equations.
(b) When is it true that for each $\underline{x} \in F^m$ there is $\underline{v} \in V$ such that for all i, $\varphi_i(\underline{v}) = x_i$?

M3 Let U, V be finite-dimensional vector spaces and let $L \in \mathrm{Hom}_F(U,V)$. Consider the pairing $V' \times U \to F$ given by $\langle \varphi, \underline{u} \rangle_L = \varphi(L\underline{u})$. Let $\{\underline{u}_j\} \subset U$, $\{\underline{v}_i\} \subset V$ be bases and let $\{\varphi_i\} \subset V'$ be the basis dual to $\{\underline{v}_i\}$. Show that the matrix of L as a linear map $U \to V$ is the same as the Gram matrix of the pairing $\langle \cdot, \cdot \rangle_L$.

Example of linear functionals: Banach limits

Let $\ell^\infty \subset \mathbb{R}^\mathbb{N}$ denote the set of *bounded* sequences (the sequences $\underline{a}$ such that for some M we have $|a_i| \le M$ for all i). Let $S\colon \mathbb{R}^\mathbb{N} \to \mathbb{R}^\mathbb{N}$ be the *shift* map $(S\underline{a})_n = \underline{a}_{n+1}$. A subspace $U \subset \mathbb{R}^\mathbb{N}$ is *shift-invariant* if $S(U) \subset U$. If U is shift-invariant a function F with domain U is called *shift-invariant* if $F \circ S = F$ (example: the subset $c \subset \mathbb{R}^\mathbb{N}$ of convergent sequences is a shift-invariant subspace, as is the functional $\lim\colon c \to \mathbb{R}$ assigning to every sequence its limit).

Note that M4 is a practice problem!

M4 (Useful facts)

(a) Show that ℓ^∞ is a subspace of $\mathbb{R}^\mathbb{N}$.
(b) Show that $S\colon \mathbb{R}^\mathbb{N} \to \mathbb{R}^\mathbb{N}$ is linear and that $S(\ell^\infty) = \ell^\infty$.
(c) Let $U \subset \mathbb{R}^\mathbb{N}$ be a shift-invariant subspace. Show that the set $U_0 = \{S\underline{a} - \underline{a} \mid \underline{a} \in U\}$ is a subspace of U.
(d) In the case $U = \mathbb{R}^{\oplus\mathbb{N}}$ of sequences of finite support, show that $U_0 = U$.
(e) Let Z be an auxiliary vector space. Show that $F \in \mathrm{Hom}(U,Z)$ is shift-invariant iff F vanishes on U_0, equivalently if F factors through U/U_0.

1. Let $W = \ell^\infty_0 \subset \ell^\infty$ in the notation of M4(c). Let $\mathbb{1}$ be the sequences everywhere equal to 1.
   (a) Show that the sum $W + \mathbb{R}\mathbb{1} \subset \ell^\infty$ is direct and construct an S-invariant functional $\varphi\colon \ell^\infty \to \mathbb{R}$ such that $\varphi(\mathbb{1}) = 1$ (*Hint*: PS2 problem 5(b)).
   (b) (Strengthening) For $\underline{a} \in \ell^\infty$ set $\|\underline{a}\|_\infty = \sup_n |a_n|$. Show that if $\underline{a} \in W$ and $x \in \mathbb{R}$ then $\|\underline{a} + x\mathbb{1}\|_\infty \ge |x|$. (Hint: consider the average of the first N entries of the vector $\underline{a} + x\mathbb{1}$).

   SUPP Let $\varphi \in (\ell^\infty)'$ be shift-invariant, positive (if $a_i \ge 0$ for all i then $\varphi(\underline{a}) \ge 0$), and satisfy $\varphi(\mathbb{1}) = 1$. Show that $\liminf_{n\to\infty} a_n \le \varphi(\underline{a}) \le \limsup_{n\to\infty} a_n$ and conclude that the restriction of φ to c is the usual limit.

2. ("choose one") Let $\varphi \in (\ell^\infty)'$ satisfy $\varphi(\mathbb{1}) = 1$. Let $\underline{a}$ be the sequence $a_n = \frac{1+(-1)^n}{2}$.
   (a) Suppose that φ is shift-invariant. Show that $\varphi(\underline{a}) = \frac{1}{2}$.
   (b) Suppose that φ respects pointwise multiplication (if $z_n = x_n y_n$ then $\varphi(\underline{z}) = \varphi(\underline{x})\varphi(\underline{y})$). Show that $\varphi(\underline{a}) \in \{0,1\}$.

SUPP If $U \subset \mathbb{R}^{\mathbb{N}}$ then $\Sigma \in U'$ is called a *summation method* if U contains the convergent series and if Σ extends the usual functional from real analysis. We call Σ *shift-invariant* if its domain is shift-invariant and if $\Sigma(\underline{a}) = a_1 + \Sigma(S\underline{a})$.
   (a) Suppose that the geometric sequence $(x^n)_{n\geq 0}$ $(x \neq 1)$ is in the domain of some shift-invariant summation method. Show that $\Sigma\left((x^n)_{n=0}^{\infty}\right) = \frac{1}{1-x}$.
   (b) Consider next the alternating series $a_n = (-1)^{n-1}n$ $(n \geq 1)$. Calculate $\underline{a} + S\underline{a} = (I+S)\underline{a}$ and show that $(I+S)^2\underline{a}$ has finite support. Use that to prove that if the shift-invariant summation method Σ applies to this sequence then

$$\Sigma(1-2+3-4+\dots) = \frac{1}{4}.$$

   (c) Consider next the sum $1+2+3+4+\cdots$. Let $b_n = n$ and let $c_n = \begin{cases} k & n = 2k \\ 0 & n = 2k+1 \end{cases}$. Show that $\underline{b} - 4\underline{c} = \underline{a}$ and conclude that if a shift-invariant summation method can sum $\underline{a}, \underline{b}, \underline{c}$ and has $\Sigma(\underline{b}) = \Sigma(\underline{c})$ (that is $1+2+3+4+\cdots = 0+1+0+2+0+3+\cdots$) then $\Sigma(\underline{b}) = -\frac{1}{12}$.

Duality and bilinear forms

DEFINITION. Let $T \in \mathrm{Hom}_F(U,V)$ be a linear map. Define a map $T' \colon V' \to U'$ (reverse direction!) by $(T'\varphi)(\underline{u}) = \varphi(T\underline{u})$ for each $\varphi \in V'$.

3. Check that T' is linear, and that the map $T \to T'$ is linear.
   RMK In the next problem set will check that the matrix of T in appropriate bases is the transpose of the matrix of T, and that $(TS)' = S'T'$ when TS are composable, explaining the analgoous formula for transposes.

4. Let $C_c^\infty(\mathbb{R})$ be the space of compactly supported smooth functions on $\mathbb{R}$ (that is, functions which have derivatives of all orders and which are identically zero outside some interval), and let $D \colon C_c^\infty(\mathbb{R}) \to C_c^\infty(\mathbb{R})$ be the differentiation operator $\frac{\mathrm{d}}{\mathrm{d}x}$. For a reasonable function f on $\mathbb{R}$ define a functional φ_f on $C_c^\infty(\mathbb{R})$ by $\varphi_f(g) = \int_{\mathbb{R}} fg\,\mathrm{d}x$ (note that f need only be integrable, not continuous).
   (a) Show that if f is continuously differentiable then $D'\varphi_f = \varphi_{-Df} = -\varphi_{Df}$. (*Hint*: this encodes a basic fact from calculus)
   DEF For this reason one usually *extends* the operator D to the dual space by $D\varphi \stackrel{\text{def}}{=} -D'\varphi$, thus giving a notion of a "derivative" for non-differentiable and even discontinuous functions.
   (b) Let the "Dirac delta" $\delta \in C_c^\infty(\mathbb{R})'$ be the evaluation functional $\delta(f) = f(0)$. Express $(D\delta)(f)$ in terms of f.
   (c) Let φ be a linear functional such that $D'\varphi = 0$. Show that for some constant c, $\varphi = \varphi_{c\mathbb{1}}$.

Supplement

A. Let V be a vector space of positive dimension over an infinite field. Let $\{V_i\}_{i=1}^{r}$ be a finite set of subspaces of V. Show that if $\bigcup_{i=1}^{r} V_i = V$ then $V_i = V$ for some i. In other words, a vector space is not a finite union of proper subspaces.

B. (This is a mostly a problem in analysis) Let $\varphi \in C_c^\infty(\mathbb{R})'$.
DEF Let $U \subset \mathbb{R}$ be open. Say that φ is *supported away from* U if for any $f \in C_c^\infty(U)$, $\varphi(f) = 0$.
The *support* $\mathrm{supp}(\varphi)$ is the complement the union of all such U.
(a) Show that $\mathrm{supp}(\varphi)$ is closed, and that φ is supported away from $\mathbb{R} \setminus \mathrm{supp}(\varphi)$.
(b) Show that $\mathrm{supp}(\delta) = \{0\}$ (see problem 4(b)).
(c) Show that $\mathrm{supp}(D\varphi) \subset \mathrm{supp}(\varphi)$ (note that this is well-known for functions).
(d) Show that $D\delta$ is not of the form φ_f for any function f.
(e) Find a (discontinuous) function θ such that $D\varphi_\theta = \delta$ as functionals on $C_c^\infty(\mathbb{R})$.