

Lior Silberman's Math 412: Problem Set 4 (due 6/2/2025)

Practice

- M1. Let U, V be vector spaces and let $U_1 \subset U, V_1 \subset V$ be subspaces.
- “Naturally” embed $U_1 \otimes V_1$ in $U \otimes V$.
 - Is $(U \otimes V) / (U_1 \otimes V_1)$ isomorphic to $(U/U_1) \otimes (V/V_1)$?
- M2. Verify the claim from class that there is a linear bijection $\{\text{bilinear forms } U \times V \rightarrow F\} \xleftrightarrow{1:1} \text{Hom}_F(U, V')$.
- M3. Let (\cdot, \cdot) be a non-degenerate bilinear form on a finite-dimensional vector space U , defined by the isomorphism $g: U \rightarrow U'$ via $(\underline{u}, \underline{v}) \stackrel{\text{def}}{=} (g\underline{u})(\underline{v})$.
- The *adjoint* of $T \in \text{End}(U)$ is the map $T^\dagger = g^{-1}T'g$ where T' is the dual map. Show that $T^\dagger \in \text{End}(U)$ satisfies $(\underline{u}, T\underline{v}) = (T^\dagger\underline{u}, \underline{v})$ for all $\underline{u}, \underline{v} \in U$.
 - Show that $(TS)^\dagger = S^\dagger T^\dagger$.
 - Show that the matrix of T^\dagger wrt an (\cdot, \cdot) -orthonormal basis is the transpose of the matrix of T in that basis.

The dual map

- Let U, V, W be vector spaces, and let $T \in \text{Hom}(U, V)$, and let $S \in \text{Hom}(V, W)$.
 - (The abstract meaning of transpose) Suppose U, V be finite-dimensional with bases $\{\underline{u}_j\}_{j=1}^m \subset U, \{\underline{v}_i\}_{i=1}^n \subset V$, and let $A \in M_{n,m}(F)$ be the matrix of T in those bases. Show that the matrix of the dual map $T' \in \text{Hom}(V', U')$ with respect to the dual bases $\{\underline{u}'_j\}_{j=1}^m \subset U', \{\underline{v}'_i\}_{i=1}^n \subset V'$ is the transpose tA .
 - Show that $(ST)^\dagger = T^\dagger S^\dagger$. It follows that ${}^t(AB) = {}^tB{}^tA$.

Bilinear forms

In problems 2,3 we assume 2 is invertible in F and fix F -vector spaces V, W .

- (Alternating pairings and symplectic forms) Let V, W be vector spaces, and let $[\cdot, \cdot]: V \times V \rightarrow W$ be a bilinear map.
 - Show that $(\forall \underline{u}, \underline{v} \in V : [\underline{u}, \underline{v}] = -[\underline{v}, \underline{u}]) \leftrightarrow (\forall \underline{u} \in V : [\underline{u}, \underline{u}] = 0)$ (Hint: consider $\underline{u} + \underline{v}$).
 DEF A form satisfying either property is *alternating*. We now suppose $[\cdot, \cdot]$ is alternating.
 PRAC Show that the *radical* $R = \{\underline{u} \in V \mid \forall \underline{v} \in V : [\underline{u}, \underline{v}] = 0\}$ of the form is a subspace.
 - The form $[\cdot, \cdot]$ is called *non-degenerate* if its radical is $\{0\}$. Show that setting $[\underline{u} + R, \underline{v} + R] \stackrel{\text{def}}{=} [\underline{u}, \underline{v}]$ defines a non-degenerate alternating bilinear map $(V/R) \times (V/R) \rightarrow W$.
 RMK Note that you need to justify each claim, starting with “defines”.
- (Darboux's Theorem) Suppose now that V is finite-dimensional, and that $[\cdot, \cdot]: V \times V \rightarrow F$ is a non-degenerate alternating form.

DEF The *orthogonal complement* of a subspace $U \subset V$ is a set $U^\perp = \{\underline{v} \in V \mid \forall \underline{u} \in U : [\underline{u}, \underline{v}] = 0\}$.

PRAC Show that U^\perp is a subspace of V .

 - Show that the restriction of $[\cdot, \cdot]$ to U is non-degenerate iff $U \cap U^\perp = \{0\}$.

- (*b) Suppose that the conditions of (a) hold. Show that $V = U \oplus U^\perp$, and that the restriction of $[\cdot, \cdot]$ to U^\perp is non-degenerate.
- (c) Let $\underline{u} \in V$ be non-zero. Show that there is $\underline{u}' \in V$ such that $[\underline{u}, \underline{u}'] \neq 0$. Find a basis $\{\underline{u}_1, \underline{v}_1\}$ to $U = \text{Span}\{\underline{u}, \underline{u}'\}$ in which the matrix of $[\cdot, \cdot]$ is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.
- (d) Show that $\dim_F V = 2n$ for some n , and that V has a basis $\{\underline{u}_i, \underline{v}_i\}_{i=1}^n$ in which the matrix of $[\cdot, \cdot]$ is block-diagonal, with each 2×2 block of the form from (d).
- RECAP Only even-dimensional spaces have non-degenerate alternating forms, and up to choice of basis, there is only one such form.

Tensor products

4. For finite-dimensional U, V construct a natural isomorphism $\text{End}(U \otimes V) \rightarrow \text{Hom}(U, U \otimes \text{End}(V))$.
 SUPP Generalize this to a natural isomorphism $\text{Hom}(U \otimes V_1, U \otimes V_2) \rightarrow \text{Hom}(U, U \otimes \text{Hom}(V_1, V_2))$.
5. Let U, W be vector spaces with U finite-dimensional, and let $A \in \text{Hom}(U, U \otimes W)$. Given a basis $\{\underline{u}_j\}_{j=1}^{\dim U}$ of U let $\underline{w}_{ij} \in W$ be defined by $A\underline{u}_j = \sum_{i=1}^{\dim U} \underline{u}_i \otimes \underline{w}_{ij}$ and define $\text{Tr}_U A = \sum_{i=1}^{\dim U} \underline{w}_{ii} \in W$. Show that this definition is independent of the choice of basis.
6. (Partial traces) Let U, V be real vector spaces equipped with non-degenerate inner products.
- (a) Show that $\langle u_1 \otimes v_1, u_2 \otimes v_2 \rangle_{U \otimes V} \stackrel{\text{def}}{=} \langle u_1, u_2 \rangle_U \langle v_1, v_2 \rangle_V$ induces an inner product on $U \otimes V$.
- (b) Let $A \in \text{End}(U), B \in \text{End}(V)$. Show that $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$ (A^\dagger, B^\dagger are defined in M2).
- (c) Let $P \in \text{End}(U \otimes V)$, interpreted as an element of $\text{Hom}(U, U \otimes \text{End}(V))$ as in 4. Show that $(\text{Tr}_U P)^\dagger = \text{Tr}_U (P^\dagger)$.
- (*d) [Thanks to J. Karczarek] Let $\underline{w} \in U \otimes V$ be non-zero, and let $P_{\underline{w}} \in \text{End}(U \otimes V)$ be the orthogonal projection on \underline{w} . It follows from (c) that $\text{Tr}_U P_{\underline{w}} \in \text{End}(V)$ and $\text{Tr}_V P_{\underline{w}} \in \text{End}(U)$ are both selfadjoint (see M3 above). Show that their non-zero eigenvalues are the same.

Supplementary problems

- A. (Extension of scalars) Let $F \subset K$ be fields. Let V be an F -vectorspace.
- (a) Considering K as an F -vectorspace (see PS1), we have the tensor product $V_K = K \otimes_F V$ (the subscript means “tensor product as F -vectorspaces”). For each $\alpha \in K$ set $\alpha(x \otimes v) \stackrel{\text{def}}{=} (\alpha x) \otimes v$. Show that this extends to an F -linear map $K \otimes_F V \rightarrow K \otimes_F V$ giving V_K the structure of a K -vector space. This construction is called “extension of scalars”
- (b) Let $\{v_i\}_{i \in I} \subset V$ be a set of vectors. Show that it is linearly independent (resp. spanning) iff $\{1_K \otimes v_i\}_{i \in I} \subset V_K$ is linearly independent (resp. spanning). Conclude that $\dim_K(K \otimes_F V) = \dim_F V$.
- (c) Let $V_N = \text{Span}_{\mathbb{R}} \left(\{1\} \cup \{\cos(nx), \sin(nx)\}_{n=1}^N \right)$. Then $\frac{d}{dx}: V_N \rightarrow V_N$ is not diagonal. Find a different basis for $\mathbb{C} \otimes_{\mathbb{R}} V_N$ in which $\frac{d}{dx}$ is diagonal. Note that the elements of your basis are not “pure tensors”, that is not of the form $af(x)$ where $a \in \mathbb{C}$ and $f = \cos(nx)$ or $f = \sin(nx)$.
- B. DEF: An F -algebra is a triple $(A, 1_A, \times)$ such that A is an F -vector space, $(A, 0_A, 1_A, +, \times)$ is a ring, and (compatibility of structures) for any $a \in F$ and $x, y \in A$ we have $a \cdot (x \times y) = (a \cdot x) \times y = x \times (a \cdot y)$. Because of the compatibility from now on we won't distinguish the multiplication in A and scalar multiplication by elements of F .
- (a) Verify that \mathbb{C} is an \mathbb{R} -algebra, and that $M_n(F)$ is an F -algebra for all F .
- (b) More generally, verify that if R is a ring, and $F \subset R$ is a subfield then R has the structure of an F -algebra. Similarly, that $\text{End}_F(V)$ is an F -algebra for any vector space V .
- (c) Let A, B be F -algebras. Give $A \otimes_F B$ the structure of an F -algebra.
- (d) Show that the map $F \rightarrow A$ given by $a \mapsto a \cdot 1_A$ gives an embedding of F -algebras $F \hookrightarrow A$.
- (e) (Extension of scalars for algebras) Let K be an extension of F . Give $K \otimes_F A$ the structure of a K -algebra.
- (f) Show that for V finite-dimensional, $K \otimes_F \text{End}_F(V) \simeq \text{End}_K(K \otimes_F V)$.
- C. The *center* $Z(A)$ of a ring is the set of elements that commute with the whole ring.
- (a) Show that the center of an F -algebra is an F -subspace, containing the subspace $F \cdot 1_A$.
- (b) Show that the image of $Z(A) \otimes Z(B)$ in $A \otimes B$ is exactly the center of that algebra.