

**Lior Silberman's Math 412: Problem Set 5 (due 13/2/2025)**

**Practice**

- P1. Let  $U = \text{Span}_F \{u_1, u_2\}$  be two-dimensional. Show that the element  $\underline{u}_1 \otimes \underline{u}_1 + \underline{u}_2 \otimes \underline{u}_2 \in U \otimes U$  is not a pure tensor, that is not of the form  $\underline{u} \otimes \underline{v}$  for any  $\underline{u}, \underline{v} \in U$ .
- P2. Let  $\iota: U \times V \rightarrow U \otimes V$  be the standard inclusion map ( $\iota(\underline{u}, \underline{v}) = \underline{u} \otimes \underline{v}$ ). Show that  $\iota(\underline{u}, \underline{v}) = 0$  iff  $\underline{u} = \underline{0}_U$  or  $\underline{v} = \underline{0}_V$  and that for non-zero vectors we have  $\iota(\underline{u}, \underline{v}) = \iota(\underline{u}', \underline{v}')$  iff  $\underline{u}' = \alpha \underline{u}$  and  $\underline{v}' = \alpha^{-1} \underline{v}$  for some  $\alpha \in F^\times$ .
- P3. Let  $U, V$  be finite-dimensional spaces and let  $A \in \text{End}(U), B \in \text{End}(V)$ .
- Construct a map  $A \oplus B \in \text{End}_F(U \oplus V)$  restricting to  $A, B$  on the images of  $U, V$  in  $U \oplus V$ .
  - Show that  $\text{Tr}(A \oplus B) = \text{Tr}(A) + \text{Tr}(B)$ .
  - Evaluate  $\det(A \oplus B)$ .

**The trace**

- Let  $U$  be finite-dimensional.
  - Construct an isomorphism  $V \otimes U' \rightarrow \text{Hom}_F(U, V)$ .
  - Define a map  $\text{Tr}: U \otimes U' \rightarrow F$  extending the evaluation pairing  $U \times U' \rightarrow F$ .

DEF The *trace* of  $T \in \text{Hom}_F(U, U)$  is given by identifying  $T$  with an element of  $U \otimes U'$  via (a) and then applying the map of (b).

  - Let  $T \in \text{End}_F(U)$ , and let  $A$  be the matrix of  $T$  with respect to the basis  $\{u_i\}_{i=1}^n \subset U$ . Show that  $\text{Tr} T = \sum_{i=1}^n A_{ii}$ .

RMK This shows that similar matrices have the same trace!

  - Solve P3(b) from this point of view.

**Tensor products of maps**

- Let  $U, V$  be finite-dimensional spaces, and let  $A \in \text{End}(U), B \in \text{End}(V)$ .
  - Show that  $(\underline{u}, \underline{v}) \mapsto (A\underline{u}) \otimes (B\underline{v})$  is bilinear, and obtain a linear map  $A \otimes B \in \text{End}(U \otimes V)$ .
  - Suppose  $A, B$  are diagonalizable. Using an appropriate basis for  $U \otimes V$ , Obtain a formula for  $\det(A \otimes B)$  in terms of  $\det(A)$  and  $\det(B)$ .
  - Extending (a) by induction, show for any  $A \in \text{End}_F(V)$ , the map  $A^{\otimes k}$  induces maps  $\text{Sym}^k A \in \text{End}(\text{Sym}^k V)$  and  $\wedge^k A \in \text{End}(\wedge^k V)$ .
  - (\*d) Show that the formula of (b) holds for all  $A, B$ .
- Suppose  $\frac{1}{2} \in F$ , and let  $U$  be finite-dimensional. Construct isomorphisms
$$\{\text{symmetric bilinear forms on } U\} \leftrightarrow (\text{Sym}^2 U)' \leftrightarrow \text{Sym}^2(U').$$

### Extension of scalars

4. (extension of scalars for linear maps) Let  $K/F$  be an extension of fields. For  $T \in \text{Hom}_F(U, V)$  let  $T_K = \text{Id}_K \otimes_F T \in \text{Hom}_F(U_K, V_K)$  as in 2(a).
- Show that  $T_K \in \text{Hom}_K(U_K, V_K)$  (i.e. that it is actually  $K$ -linear not only  $F$ -linear).
  - (Functoriality) Show that  $\text{Id}_{U_K} = (\text{Id}_U)_K$ . For  $S \in \text{Hom}_F(V, W)$ . Show that  $(S \circ T)_K = S_K \circ T_K$ .
  - (Linear algebra) If  $U \subset V$  we identify  $U_K$  with a subspace of  $V_K$  via the inclusion map. Show that (with this identification) we have  $\text{Ker } T_K = (\text{Ker } T)_K$  and  $\text{Im } T_K = (\text{Im } T)_K$ .
  - Let  $B_U \subset U$  be an  $F$ -basis. Show that  $\{1_K \otimes \underline{u}\}_{\underline{u} \in B_U} \subset U_K$  is a  $K$ -basis of  $U_K$ .
  - Let  $B_U, B_V$  be bases of  $U, V$  respectively. Show that the matrix of  $T_K$  with respect to the corresponding bases of  $U_K, V_K$  is the same as the matrix of  $T$  with respect to the original bases.
5. (extension of scalars and constructions) Construct “natural” isomorphisms:
- $\bigoplus_{i \in I} (V_i)_K \rightarrow (\bigoplus_{i \in I} V_i)_K$
  - $U_K/V_K \rightarrow (U/V)_K$ .
  - $U_K \otimes_K V_K \rightarrow (U \otimes_F V)_K$ .
- HINT in each case show that both sides satisfy the appropriate universal property for  $K$ -vectorspaces.
- (\*d) Show that the natural map  $(\prod_{i \in I} V_i)_K \rightarrow \prod_{i \in I} (V_i)_K$  is, in general, not surjective.

### Supplement: Nilpotence

- A. Let  $U \in M_n(F)$  be *strictly upper-triangular*, that is upper triangular with zeroes along the diagonal. Show that  $U^n = 0$  and construct such  $U$  with  $U^{n-1} \neq 0$ .

Fix a vector space  $V$  and a linear map  $T \in \text{End}(V)$ .

DEFINITION. Call  $T$  *locally nilpotent* if  $\forall \underline{v} \in V : \exists k \geq 0 : T^k \underline{v} = \underline{0}$ ; call  $T$  *nilpotent* if  $\exists k \geq 0 : \forall \underline{v} \in V : T^k \underline{v} = \underline{0}$ .

- B. (a) Show that every nilpotent map is locally nilpotent.  
 (b) Suppose  $T$  is locally nilpotent and  $V$  is finite-dimensional. Show that  $T$  is nilpotent.  
 (c) Show that  $T$  is locally nilpotent iff for every  $\underline{v} \in V$  there is a finite-dimensional  $T$ -invariant subspace  $\underline{w} \in W \subset V$  such that  $T|_W$  is nilpotent.  
 (d) For any infinite-dimensional  $V$  find an example of  $T \in \text{End}(V)$  which is locally nilpotent but not nilpotent.  
 (\*e) Suppose  $T$  is nilpotent. Show that  $I + T$  is invertible.  
 (\*\*f) Suppose  $T$  is locally nilpotent. Show that  $I + T$  is invertible.
- C. (a) Find nilpotent  $A, B \in M_2(F)$  such that  $A + B$  isn't nilpotent.  
 (c) Suppose that  $A, B \in \text{End}(V)$  are nilpotent and that  $A, B$  commute. Show that  $A + B$  is nilpotent.

### Supplement: the tensor algebra

- D. (The tensor algebra) Fix a vector space  $U$ .
- (a) Extend the bilinear map  $\otimes: U^{\otimes n} \times U^{\otimes m} \rightarrow U^{\otimes n} \otimes U^{\otimes m} \simeq U^{\otimes(n+m)}$  to a bilinear map  $\otimes: \bigoplus_{n=0}^{\infty} U^{\otimes n} \times \bigoplus_{n=0}^{\infty} U^{\otimes n} \rightarrow \bigoplus_{n=0}^{\infty} U^{\otimes n}$ .
- (b) Show that this map  $\otimes$  is associative and distributive over addition. Show that  $1_F \in F \simeq U^{\otimes 0}$  is an identity for this multiplication.
- DEF This algebra is called the *tensor algebra*  $T(U)$ .
- (c) Show that the tensor algebra is *free*: for any  $F$ -algebra  $A$  and any  $F$ -linear map  $f: U \rightarrow A$  there is a unique  $F$ -algebra homomorphism  $\bar{f}: T(U) \rightarrow A$  whose restriction to  $U^{\otimes 1}$  is  $f$ .

- E. (The symmetric algebra). Fix a vector space  $U$ .
- (a) Endow  $\bigoplus_{n=0}^{\infty} \text{Sym}^n U$  with a product structure as in 3(a).
- (b) Show that this creates a commutative algebra  $\text{Sym}(U)$ .
- (c) Fixing a basis  $\{\underline{u}_i\}_{i \in I} \subset U$ , construct an isomorphism  $F[\{x_i\}_{i \in I}] \rightarrow \text{Sym}^* U$ .
- RMK In particular,  $\text{Sym}^*(U')$  gives a coordinate-free notion of “polynomial function on  $U$ ”.
- (d) Let  $I \triangleleft T(U)$  be the two-sided ideal generated by all elements of the form  $\underline{u} \otimes \underline{v} - \underline{v} \otimes \underline{u} \in U^{\otimes 2}$ . Show that the map  $\text{Sym}(U) \rightarrow T(U)/I$  is an isomorphism.

RMK When the field  $F$  has finite characteristic, the correct definition of the symmetric algebra (the definition which gives the universal property) is  $\text{Sym}(U) \stackrel{\text{def}}{=} T(U)/I$ , not the space of symmetric tensors.

### Supplement: local finiteness

- F. Let  $V$  be a (possibly infinite-dimensional) vector space,  $A \in \text{End}(V)$ .
- (a) Show that the following are equivalent for  $\underline{v} \in V$ :
- (1)  $\dim_F \text{Span}_F \{A^n \underline{v}\}_{n=0}^{\infty} < \infty$ ;
  - (2) there is a finite-dimensional subspace  $W \subset V$  such that  $AW \subset W$ .
- DEF Call such  $\underline{v}$  *locally finite*, and let  $V_{\text{fin}}$  be the set of locally finite vectors.
- (b) Show that  $V_{\text{fin}}$  is a subspace of  $V$ .
- (c) Call  $A$  *locally nilpotent* if for every  $\underline{v} \in V$  there is  $n \geq 0$  such that  $A^n \underline{v} = \underline{0}$  (condition (1) of 5(a)). Find a vector space  $V$  and a locally nilpotent map  $A \in \text{End}(V)$  which is not nilpotent.
- (\*d)  $A$  is called *locally finite* if  $V_{\text{fin}} = V$ , that is if every vector is contained in a finite-dimensional  $A$ -invariant subspace. Find a space  $V$  and locally finite linear maps  $A, B \in \text{End}(V)$  such that  $A + B$  is not locally finite.