

Lior Silberman's Math 412: Problem Set 6 (due 4/3/2025)

M1. (Minimal polynomials)

$$\text{Let } A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -2 & -2 & 2 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix}.$$

- (a) Find the minimal polynomial of A and show that the minimal polynomial of B is $x^2(x-1)^2$.
- (b) Find a 3×3 matrix whose minimal polynomial is x^2 .

M2. For each of A, B find its eigenvalues and the corresponding generalized eigenspaces.

Triangular matrices

- 1. Let U be an upper-triangular square matrix with non-zero diagonal entries.
 - (a) Give a “backward-substitution” algorithm for solving $U\mathbf{x} = \mathbf{b}$ efficiently.
 - (b) Explicitly use your algorithm to solve $\begin{pmatrix} 1 & 4 & 5 \\ & 2 & 6 \\ & & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$.
 - (c) For a general upper-triangular U give a formula for U^{-1} , proving in particular that U is invertible and that U^{-1} is again upper-triangular.

RMK It's possible to show that if $\mathcal{A} \subset M_n(F)$ is a subspace containing the identity matrix and closed under matrix multiplication, then the inverse of any matrix in \mathcal{A} belongs to \mathcal{A} . This applies, in particular, to the set of upper-triangular matrices.

The minimal polynomial

- 2. Let $D \in M_n(F) = \text{diag}(a_1, \dots, a_n)$ be diagonal.
 - (a) For any polynomial $p \in F[x]$ show that $p(D) = \text{diag}(p(a_1), \dots, p(a_n))$.
 - (b) Show that the minimal polynomial of D is $m_D(x) = \prod_{j=1}^r (x - a_{i_j})$ where $\{a_{i_j}\}_{j=1}^r$ is an enumeration of the distinct values among the a_i .
 - (c) Show that (over any field) the matrix B from problem P1 is not similar to a diagonal matrix.
 - (d) Now suppose that U is an upper-triangular matrix with diagonal D . Show that for any $p \in F[x]$, $p(U)$ has diagonal $p(D)$. In particular, $m_D | m_U$.
 - 3. Let $T \in \text{End}(V)$ be diagonalizable. Show that every generalized eigenspace is simply an eigenspace.
 - 4. Let $S \in \text{End}(U)$, $T \in \text{End}(V)$. Let $S \oplus T \in \text{End}(U \oplus V)$ be the “block-diagonal map”.
 - (a) For $f \in F[x]$ show that $f(S \oplus T) = f(S) \oplus f(T)$.
 - (b) Show that $m_{S \oplus T} = \text{lcm}(m_S, m_T)$ (“least common multiple”: the polynomial of smallest degree which is a multiple of both).
 - (c) Conclude that $\text{Spec}_F(S \oplus T) = \text{Spec}_F(S) \cup \text{Spec}_F(T)$.
- RMK See also problem A below.

5. Let K/F be an extension of fields, let V be a finite-dimensional F -vectorpsace and let $T \in \text{End}_F(V)$. Show that the minimal and characteristic polynomials of $T_K \in \text{End}_K(V_K)$ are identical with those of T .

Supplementary problems

- A. Let $R \in \text{End}(U \oplus V)$ be “block-upper-triangular”, in that $R(U) \subset U$.
- Define a “quotient linear map” $\bar{R} \in \text{End}(U \oplus V/U)$.
 - Let S be the restriction of R to U . Show that both $m_S, m_{\bar{R}}$ divide m_R .
 - Let $f = \text{lcm}[m_S, m_{\bar{R}}]$ and set $T = f(R)$. Show that $T(U) = \{\underline{0}\}$ and that $T(V) \subset U$.
 - Show that $T^2 = 0$ and conclude that $f \mid m_R \mid f^2$.
 - Show that $\text{Spec}_F(R) = \text{Spec}_F(S) \cup \text{Spec}_F(\bar{R})$.
- B. (Cholesky decomposition)
- Let A be a positive-definite square matrix. Show that $A = LL^\dagger$ for a unique lower-triangular matrix L with positive entries on the diagonal.
- DEF For $\varepsilon \in \pm 1$ define $D^\varepsilon \in M_n(\mathbb{R})$ by $D_{ij}^\varepsilon = \begin{cases} \varepsilon & j = i + \varepsilon \\ -\varepsilon & j = i \\ 0 & j \neq i, i + \varepsilon \end{cases}$ and let $A = -D^- D^+$ be the
- (positive) discrete Laplace operator.
- To $f \in C^\infty(0, 1)$ associate the vector $\underline{f} \in \mathbb{R}^n$ where $\underline{f}(i) = f(\frac{i}{n})$. Show that $\frac{1}{n}D^+ \underline{f}$ and $\frac{1}{n}D^- \underline{f}$ are both close to \underline{f}' (so that both are discrete differentiation operators). Show that $\frac{1}{n^2}D^- D^+$ is an approximation to the second derivative.
 - Find a lower-triangular matrix L such that $LL^\dagger = A$.
- C. (Rational canonical form part I) Let $T \in \text{End}(V)$. For monic irreducible $p \in F[x]$ define $V_p = \{\underline{v} \in V \mid \exists k : p(T)^k \underline{v} = \underline{0}\}$.
- Show that V_p is a T -invariant subspace of V and that $m_{T|_{V_p}} = p^k$ for some $k \geq 0$, with $k \geq 1$ iff $V_p \neq \{\underline{0}\}$. Conclude that $p^k \mid m_T$.
 - Show that if $\{p_i\}_{i=1}^r \subset F[x]$ are distinct monic irreducibles then the sum $\bigoplus_{i=1}^r V_{p_i}$ is direct.
 - Let $\{p_i\}_{i=1}^r \subset F[x]$ be the prime factors of $m_T(x)$. Show that $V = \bigoplus_{i=1}^r V_{p_i}$.
 - Suppose that $m_T(x) = \prod_{i=1}^r p_i^{k_i}(x)$ is the prime factorization of the minimal polynomial. Show that $V_{p_i} = \text{Ker } p_i^{k_i}(T)$.