Lior Silberman's Math 412: Problem Set 7

Practice

M1. Find the characteristic and minimal polynomial of each matrix:

$$\begin{pmatrix}1&1&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\ \begin{pmatrix}5&0&0&0&0&0\\0&4&0&0&0&0\\0&0&2&0&0&0\\0&0&0&2&1&0\\0&0&0&0&2&1\\0&0&0&0&0&2\end{pmatrix},\ \begin{pmatrix}5&0&0&0&0&0\\0&2&1&0&0&0\\0&0&2&0&0&0\\0&0&0&2&1&0\\0&0&0&0&2&1\\0&0&0&0&0&2\end{pmatrix}.$$

M2. Show that $\begin{pmatrix}0&1&\alpha\\0&0&1\\0&0&0\end{pmatrix}$, $\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}$ are similar. Generalize to higher dimensions.

Minimal polynomials

1. Let $T,S \in \mathrm{End}_F(V)$.
   (a) Suppose that T,S are similar. Show that $m_T(x) = m_S(x)$.
   (b) Prove or disprove: if $m_T(x) = m_S(x)$ and $p_T(x) = p_S(x)$ then T,S are similar.

2. Let V be finite-dimensional, and let $\mathcal{A} \subset \mathrm{End}_F(V)$ be an "F-subalgebra", that is a subspace containing the identity map and closed under multiplication (composition). Suppose that $T \in \mathcal{A}$ is invertible in $\mathrm{End}_F(V)$. Show that $T^{-1} \in \mathcal{A}$.

3. (The rational canonical form 1) In this problem we describe a direct sum decomposition that is appropriate for fields F that are not necessarily algebraically closed. For this we generalize the notion of an eigenvalue: for monic irreducible $p \in F[x]$ define $V_p = \left\{\underline{v} \in V \mid \exists k : p(T)^k \underline{v} = \underline{0}\right\}$.
   (a) Show that V_p is a T-invariant subspace of V and that $m_{T\restriction V_p} = p^k$ for some $k \geq 0$, with $k \geq 1$ iff $V_p \neq \{\underline{0}\}$. Conclude that $p^k | m_T$.
   (b) Let $f \in F[x]$. Show that the restriction $f(T) \restriction V_p$ is invertible iff $p \nmid f$.
   (c) Show that if $\{p_i\}_{i=1}^r \subset F[x]$ are distinct monic irreducibles then the sum $\bigoplus_{i=1}^r V_{p_i}$ is direct.
   (d) Let $\{p_i\}_{i=1}^r \subset F[x]$ be the prime factors of $m_T(x)$. Show that $V = \bigoplus_{i=1}^r V_{p_i}$.
   (e) Suppose that $m_T(x) = \prod_{i=1}^r p_i^{k_i}(x)$ is the prime factorization of the minimal polynomial. Show that $V_{p_i} = \mathrm{Ker}\, p_i^{k_i}(T)$ and that $m_{T\restriction V_{p_i}} = p_i^{k_i}$.

Supplementary problem

We continue problem 3. By the direct sum decomposition we can concentrate on a single V_{p}., so from now on we may assume $m_T(x) = p(x)^k$ with p irreducible of degree d.

A. Let $K = F[x]/(p(x))$ (quotient of the ring $F[x]$ by the ideal $pF[x]$).

(a) Show that K is a field, generated over K by the image of the polynomial $x \in F[x]$ (write β for this image, so that $K = F(\beta)$).

(b) Let W be an F-vector space and $T \in \mathrm{End}_F(W)$ with minimal polynomial p. Show that W has the structure of a K-vector space where multiplication by elements of F still has the same meaning and such that $\beta \underline{w} = T\underline{w}$.
Hint: for $f \in F[x]$ set $(f + (p(x)) \cdot \underline{w} = f(T)\underline{w}$. Start by checking that this is well-defined (independent of the choice of f).

– Recall now that we are working in a vector space V such that $p(T)^k = 0$ for some k.

(c) Let $W_i = \mathrm{Im}(p(T)^i) \cap \mathrm{Ker}(p(T))$. Show that $\{0\} = W_k \subset \cdots \subset W_0 = \mathrm{Ker}(p(T))$ and that each W_i is a K-subspace of W_0 for the vector space structure of 2(b).