## Lior Silberman's Math 412: Problem Set 10

P1. Recall that a *projection* is a linear map $E$ such that $E^2 = E$. For each $n$ construct a projection $E_n \colon \mathbb{R}^2 \to \mathbb{R}^2$ of norm at least $n$ ($\mathbb{R}^n$ is equipped with the Euclidean norm unless specified otherwise). Prove for yourself that the norm of an *orthogonal* projection is 1.

### Difference and Differential Equations

P2. Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Let $\underline{v}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

(a) Find $S$ invertible and $D$ diagonal such that $A = S^{-1}DS$.
(b) Verify that $A^k = S^{-1}D^kS$.
(c) Find a formula for $\underline{v}_k = A^k\underline{v}_0$, and show that $\frac{\underline{v}_k}{\|\underline{v}_k\|}$ converges for any norm on $\mathbb{R}^2$.

RMK You have found a formula for Fibbonacci numbers (why?), and have shown that the real number $\frac{1}{2}\left(\frac{1+\sqrt{5}}{2}\right)^n$ is exponentially close to being an integer.

RMK This idea can solve any *difference equation* (see PS10). We now apply this to solving *differential* equations.

1. We will analyze the differential equation $u'' = -u$ with initial data $u(0) = u_0$, $u'(0) = u_1$.
   (a) Let $\underline{v}(t) = \begin{pmatrix} u(t) \\ u'(t) \end{pmatrix}$. Show that $u$ is a solution to the given equation iff $\underline{v}(t)$ is a solution to
   $$\underline{v}'(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \underline{v}(t)\,.$$
   (b) Let $W = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Find formulas for $W^n$ and express $\exp(Wt) = \sum_{k=0}^{\infty} \frac{W^k t^k}{k!}$ as a matrix whose entries are standard power series.
   (c) Sum the series and show that $u(t) = u_0\cos(t) + u_1\sin(t)$.
   (d) Find a matrix $S$ such that $W = S\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}S^{-1}$. Evaluate $\exp(Wt)$ again, this time using $\exp(Wt) = S\left(\exp\begin{pmatrix} it & 0 \\ 0 & -it \end{pmatrix}\right)S^{-1}$.

2. Consider the differential equation $\frac{\mathrm{d}}{\mathrm{d}t}\underline{v} = B\underline{v}$ where $B$ is at in PS7 problem 1.
   (a) Find matrices $S, D$ so that $D$ is in Jordan form, and such that $B = SDS^{-1}$.
   (b) Find $\exp(tD)$ as in 1(b) by computing a formula for $D^n$ and summing the series.
   (c) Find the solution such that $\underline{v}(0) = \begin{pmatrix} 0 & 1 & 1 & 0 \end{pmatrix}^{\mathrm{t}}$.

3. Let $A = \begin{pmatrix} z & 1 \\ 0 & z \end{pmatrix}$ with $z \in \mathbb{C}$.
   (a) Find (and prove) a simple formula for the entries of $A^n$.
   (b) Use your formula to decide the set of $z$ for which $\sum_{n=0}^{\infty} A^n$ converge, and give a formula for the sum.
   (c) Show that the sum is $(\mathrm{Id} - A)^{-1}$ when the series converges.

## Extra credit

4. For any matrix $A$ show that $\sum_{n=0}^{\infty} z^n A^n$ converges for $|z| < \frac{1}{\rho(A)}$.

## Supplementary problems

A. Consider the map $\mathrm{Tr}\colon M_n(F) \to F$.
   (a) Show that this is a continuous map.
   (b) Find the norm of this map when $M_n(F)$ is equipped with the $L^1 \to L^1$ operator norm (see PS8 Problem 2(a)).
   (c) Find the norm of this map when $M_n(F)$ is equipped with the Hilbert–Schmidt norm (see PS8 Problem 4).
   (*d) Find the norm of this map when $M_n(F)$ is equipped with the $L^p \to L^p$ operator norm. Find the matrices $A$ with operator norm 1 and trace maximal in absolute value.

B. Call $T \in \mathrm{End}_F(V)$ *bounded below* if there is $K > 0$ such that $\|T\underline{v}\| \geq K\|\underline{v}\|$ for all $\underline{v} \in V$.
   (a) Let $T$ be boudned below. Show that $T$ is invertible, and that $T^{-1}$ is a bounded operator.
   (*b) Suppose that $V$ is finite-dimensional. Show that every invertible map is bounded below.

C. (The supremum norm and the Weierestrass $M$-test) Let $V$ be a complete normed space.
   DEF For a set $X$ call $f\colon X \to V$ *bounded* if there is $M > 0$ such that $\|f(x)\|_V \leq M$ for all $x \in X$ in which case we write $\|f\|_\infty = \sup_{x\in X} \|f(x)\|_V$ (equivalently, $f$ is bounded if $x \mapsto \|f(x)\|_V$ is in $\ell^\infty(X)$).
   (a) Show that $\ell^\infty(X;V)$ is a vector space (this doesn't use completeness of $V$).
   (b) Show that $\ell^\infty(X;V)$ is complete.
   DEF Now suppose that $X$ is a topological space (if you aren't sure about this, simply assume $X \subset \mathbb{R}^n$). Let $C(X;V)$ denote the space of *continuous* functions $X \to V$ and let $C_\mathrm{b}(X;V) = C(X;V) \cap \ell^\infty(X;V)$ be the space of *bounded* continuous functions, the latter equipped with the $\ell^\infty$-norm.
   (c) Show that $C_\mathrm{b}(X;V)$ is complete (equivalently, that it is a closed subspace of $\ell^\infty(X;V)$.
   COR Deduce Weirestrass's $M$-test: $f_n\colon X \to V$ are continuous and $\|f_n\|_\infty \leq M_n$ with $\sum_n M_n < \infty$ then $\sum_n f_n$ converges to a continuous function bounded by $\sum_n M_n$.