Math 428: Problem Set 1, due 16/1/2025

- **Undergraduate students:** Problems marked SUPP are supplementary; consider them extra credit.
- **Graduate students**: Consider all problems and do a reasonable amount based on your goals for the ocurse; choose problems based on what you already know and what you need to practice.

Kinematics

1. Prove Lemma 30 from the notes: let $\gamma\colon I \to \mathbb{E}^M$ be a differentiable curve, and let $F\colon \mathbb{E}^M \to \mathbb{R}^m$ be a differentiable function. Suppose that for some $t_0 \in I$ we have $F(\gamma(t_0)) = 0$. Then $\gamma(t) \in X \stackrel{\text{def}}{=} F^{-1}(\underline{0})$ holds for all t iff for all t $\dot{\gamma}(t) \in \operatorname{Ker} dF_{\gamma(t)}$.

In the next three problems (borrowed from Dr. Joanna Karczmarek's PHYS 350 homework) give a system of coordinates on configuration space, the associated parametrization of the configuratoins, and write the kinetic and potential energies as functions of the coordinates. See diagrams on the page 3, and note that the problem set continues after the images.

2. A bead sliding without friction on a circular wire loop. The loop has radius R and is mounted at an angle with one end higher by h than the other end.
3. Fix a conical surface whose shape is given in cylindrical coordinates r, θ, z by $r = \alpha(Z - z)$ ($\alpha, Z > 0$ are fixed). At the apex of the cone there is a small hole through which a massless string is threaded. A particle of mass m is hanging from the thread inside the cone, and a particle of mass M is attached at the other end, resting on the cone's outer surface. Assume particle m does not hit the inner surface of the cone and that particle M does not leave the outer surface..
4. A compound pendulum: two particles of masses m_1, m_2 connected by rigid massless rods of length L and frictionless joints as shown in the diagram (the joint on the ceiling flexes in the plane of the page; the lower joint flexes in the plan that contains the top rod and is perpendicular to the page (of the front view).

5*. We compare examples with holonomic and non-holonomic constraints. In both problems a circular hoop of radius R is moving in $\mathbb{E}^3$, where the hoop is constrained to remain vertical and tangent to the xy-plane and to roll without slipping there. In this problem it is easier to work in coordinate space (i.e. in terms of the values of the coordinates), so e.g. in part (a) think of the curve $\gamma(t)$ as the coordinate curve $(x(t), \theta(t))$ so that $\dot{\gamma}(t) = \begin{pmatrix} \dot{x}(t) \\ \dot{\theta}(t) \end{pmatrix}$.

   (a) Suppose the hoop moves along a line (say the x-axis). We use the coordinates $(x, \theta) \in X_1 = \mathbb{R} \times S^1$ where the first coordinate represents the position of the centre of the hoop (equivalently the point where it touches the plane) and the second the position of a marked point on the hoop. The condition of *rolling without slipping* is $dx = R d\theta$ or equivalently $dx - R d\theta = 0$. In terms of a coordinate curve this means $\langle dx - R d\theta, \dot{\gamma}(t) \rangle = 0$ (pairing of dual vector and vector, concretely that $\dot{x}(t) - R\dot{\theta}(t) = 0$). Show that there is a smooth function $F\colon X_1 \to S^1$ so that curves conforming to the rolling without slipping remain on level sets of F).
   Hint: what should dF be?

(b) Suppose now the hoop now free to spin about the vertical axis, but must roll without slipping when turning around its centre. Let us use the coordinates $(x,y,\phi,\theta) \in \mathbb{R}^2 \times S^1 \times S^1$ where (x,y) is the position of the centre, ϕ is the orientation of the hoop (say the direction of the line formed by intersecting the plane of the hoop with the xy-plane), and θ is again the position of a marked point along the hoop. Our constraints on the motion are now

$$\begin{cases} dx = R\cos\phi \cdot d\theta \\ dy = R\sin\phi \cdot d\theta \end{cases}.$$

Show that these are linearly independent linear functionals on the tangent space (in coordinates these are the row vectors $(1,0,0,-R\cos\phi), (0,1,0,-R\sin\phi)$). Thus at each point x of 4d configuration space there is a 2d subspace $V_x \subset T_xX$ such that γ must be tangent to V_x at each point.

(c) Now, for any two points in configuration space, describe is a curve $\gamma(t) = (x(t), y(t), \phi(t), \theta(t))$ (you don't have to write it down explicitly, just explain in words how you would move the hoop) which connects the two configurations yet satisfies the constraints on the motion. Explain why this means that the constraint is non-holonomic – why there is no function F similar to the one in part (a).

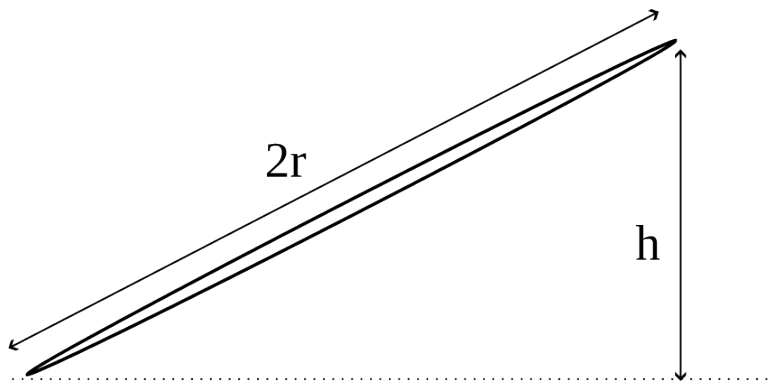
Ordinary Differential Equations

Motivation for studying differential equations: A key component of mechanics are the *equations of motion* of the system. Newton invented calculus to solve these differential equations.

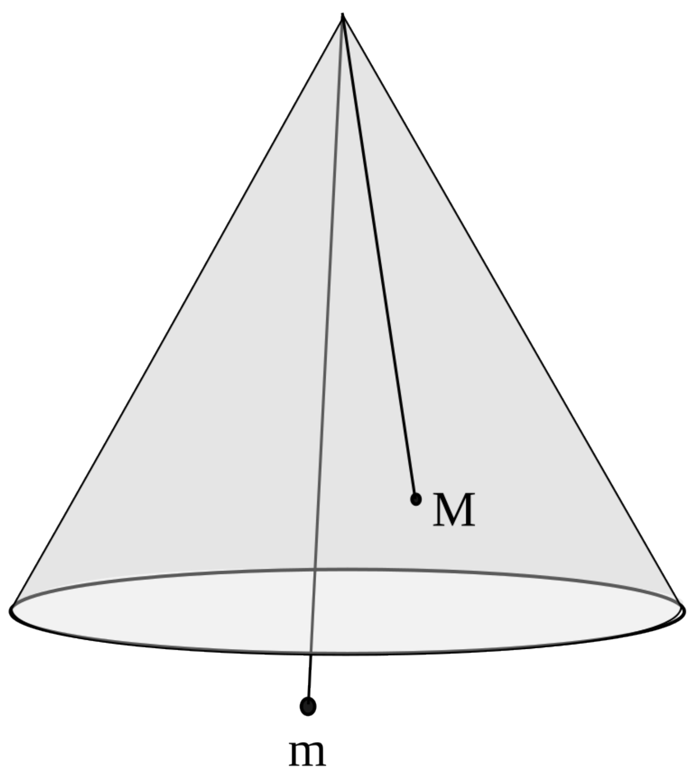
DEFINITION. Fix an open set $\Omega \subset \mathbb{R}^n$ and an interval $I \subset \mathbb{R}$. An *ordinary differential equation* is a function $F\colon I \times \Omega \to \mathbb{R}^n$. A *solution* of the differential equation is a function $\underline{y}\colon J \to \Omega$ defined on a subinterval $J \subset I$ such that for all $t_1, t_2 \in J$ we have

$$\underline{y}(t_2) - \underline{y}(t_1) = \int_{t_1}^{t_2} F\left(t; \underline{y}(t)\right) dt\,.$$

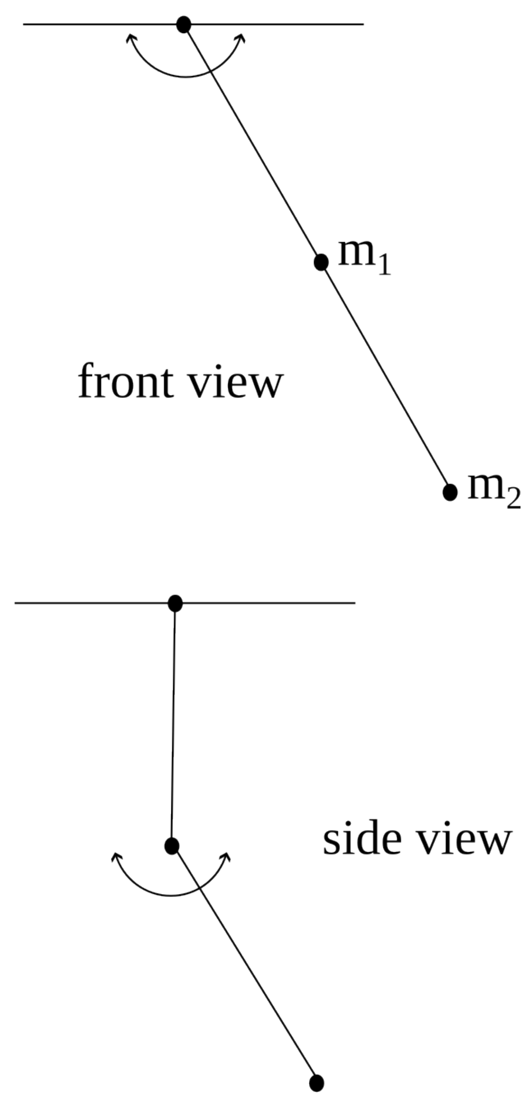
6. (Regularity) Suppose F is continuous on its domain, and that the solution y is continuous.
   (a) Show that the solution is differentiable on its interval of definition, with $\dot{\underline{y}}(t) = F\left(t; \underline{y}(t)\right)$ for all $t \in J$
   Hint: this is classical theorem from single-variable calculus.
   (b) Suppose that $F \in C^k(I \times \Omega)$, the class of k-times continuously differentiable functions. Show that the solution satisfies $\underline{y} \in C^{k+1}(J)$.
   SUPP Show that a measurable solution is, in fact, continuous. Show that this holds for measurable F with appropriate boundedness hypotheses.



(Nearly) side view of the loop for question 2.



The cone for question 3.



Compound pendulum for question 4.

Supplement: ODE

A. (the Euler Scheme) Let $t_0 \in I$ and fix an *initial condition* $\underline{y}_0 \in \Omega$.

(a) Show (as we will in class) that for some $,\varepsilon, R, M > 0$ with $D = [t_0, t_0 + \varepsilon] \times B\left(\underline{y}_0; R\right) \subset I \times \Omega$ we have $|F| \le M$ on D and $\varepsilon \le \frac{R}{M+1}$.

DEF (Euler scheme) Given $N > 0$ let $h = \frac{\varepsilon}{N}$; for $0 \le i \le N$ let $t_i = t_0 + ih$ and for $0 \le i \le N-1$ recursively set

$$\boxed{\underline{y}_i = \underline{y}_{i-1} + hF\left(t_i; \underline{y}_i\right) .}$$

Let $\underline{y}^{(N)}$ be the piecewise-linear function on $[t_0, t_0 + \varepsilon]$ interpolating the points $\left\{\left(t_i; \underline{y}_i\right)\right\}$ (so $\underline{y}_i^{(N)}(t_i) = \underline{y}_i$ and $\underline{y}^{(N)}$ is linear on $[t_i, t_{i+1}]$).

(b) Show that $\underline{y}^{(N)}$ are all valued in D, and that they are all M-Lipschitz functions.

(c) (The Peano Existence Theorem) Invoke the Arzela–Ascoli theorem to obtain a subsequential unform limit $\underline{y}(t)$ of the $\left\{\underline{y}^{(N)}\right\}_{N \ge 1}$. Show that $\underline{y}(t)$ solves the differential equation and satisfies $\underline{y}(t_0) = \underline{y}_0$.

B. (Convergence, existence, and uniqueness) Suppose that $F \in C^1$ and let $\underline{y}$ be any solution. Find a constant C such that $\left|\underline{y}^{(N)}(t_i) - \underline{y}(t_i)\right| \le Cih^2$ for all N, i. Conclude that $\left\|\underline{y}^{(N)} - \underline{y}\right\|_{C(t_0, t_0+\varepsilon)} \le \frac{C'}{N}$.

Supplement: Affine Algebra

C. (Affine subspaces) Let V be a real vector space (this problem makes sense over every field)

DEF Let $\{\underline{v}_i\}_{i=0}^r \subset V$ where $r \ge 1$. An *affine combination* of these vectors is a vector of the form

$$\sum_{i=0}^r t_i \underline{v}_i$$

where $\{t_i\}_{i=0}^r \subset \mathbb{R}$ satisfy $\sum_{i=0}^r t_i = 1$. In particular write $[\underline{v}_0, \underline{v}_1]_t = (1-t)\underline{v}_0 + t\underline{v}_1$.

DEF Say $\underline{v} \in V$ *affinely depends* on $S \subset V$ if it is an affine combination of some vectors in V.

(a) Let $A \subset V$. Show that A is closed under taking affine combinations iff it is closed under taking affine combinations of length 2.

DEF Call a nonempty $A \subset V$ satisfying the conditions of (a) an *affine subspace*. Note that V is an affine subspace of itself.

(b) Show that the intersection of affine subspaces is an affine subspace.

DEF The *affine hull* $\mathrm{aff}(S)$ of a nonempty $S \subset V$ is the intersection of all affine subsapces containing it.

(c) Let $S \subset V$ be non-empty. Show that $\mathrm{aff}(S)$ is also the set of all vectors that depend affinely on S.

D. DEF The *Minkowski sum* of $A, B \subset V$ (any subsets) is $\{\underline{a} + \underline{b} \mid \underline{a} \in A, \underline{b} \in B\}$. We write $A + \underline{w}$ for $A + \{\underline{w}\}$.

(a) Show that the Minkowski sum of two subspaces is a subspace, and of two affine subspaces is an affine subspace.

(b) Let $W \subset V$ be a subspace and let $\underline{v} \in V$. Show that $W + \underline{v}$ is an affine subspace (hint: this is a special case of (a)).

(c) Let A be an affine subspace and let $\underline{a} \in A$. Show that $A - \underline{a} = A + (-\underline{a})$ is a subspace and that $A = (A - \underline{a}) + \underline{a}$.

SUPP Show that this subspace does not depend on the choice of $\underline{a}$. Its dimension is called the *dimension* of A. Check that $\dim A = r < \infty$ if and only if A is the affine hull of a set of size $r+1$ but not of a set of size r.

E. "Affine algebra" = "linear algebra up to translation" I

FIX $\underline{z} \in V$.

(a) Let $\underline{z} \in V$ be any vector ("the origin"). Define new operations on V in terms of the original operations as follows

$$\underline{v}_1 \tilde{+} \underline{v}_2 = \underline{z} + (\underline{v}_1 - \underline{z}) + (\underline{v}_2 - \underline{z}) = \underline{v}_1 + \underline{v}_2 - \underline{z}$$
$$t \tilde{\cdot} \underline{v} = \underline{z} + t(\underline{v} - \underline{z}) = t\underline{v} + (1-t)\underline{z}.$$

Show that $\tilde{V} = (V, \tilde{+}, \tilde{\cdot})$ is also a vector space, and that $\underline{v} \mapsto \underline{v} + \underline{z}$ is an isomorphism $V \to \tilde{V}$ (hint: show the second claim first).

RMK Observe that these operations were defined only in terms of affine combinations, so they only require knowing how to make affine combinations not linear ones!

(b) Let $\{\underline{v}_i\}_{i=0}^r \subset V$ and $\{t_i\}_{i=0}^r \subset \mathbb{R}$ with $\sum_{i=0}^r t_i = 1$. Show that the affine combination defined by this data is the same in V and $\tilde{V}$.

INTERPRETATION "Affine combinations do not depend on the choice of origin".

F. "Affine algebra" = "linear algebra up to translation" II

DEF Let V, W be vector spaces. An map $f: V \to W$ is *affine if* $f\left([\underline{v}_0, \underline{v}_1]_t\right) = [f(\underline{v}_0), f(\underline{v}_1)]_t$ *for all* $\underline{v}_0, \underline{v}_1 \in V$ *and* $t \in \mathbb{R}$.

(a) Let $T \in \mathrm{Hom}_{\mathbb{R}}(V, W)$ be a linear map and let $\underline{w} \in W$ be a fixed vector. Show that $f(\underline{v}) = T\underline{v} + \underline{w}$ is an affine map.

(b) Show that every affine map has this form.

(c) Let $\mathrm{Aff}(V)$ be the group of *invertible* affine maps from V to V. Show that $\mathrm{Aff}(V)$ acts transitively on V, that the point stabilizer is isomorphic to the group $\mathrm{GL}(V)$ of invertible *linear* maps, and that $\mathrm{Aff}(V) \simeq \mathrm{GL}(V) \ltimes V$ (by V here we mean additive group acting by translation).

G. (hard) Suppose that $\dim_{\mathbb{R}} V \geq 2$ and let $f: V \to V$ be a bijection such that for every affine line $L \subset V$, $f(L)$ is also an affine line. Show that f is an affine map. Find a counterexample over another field.