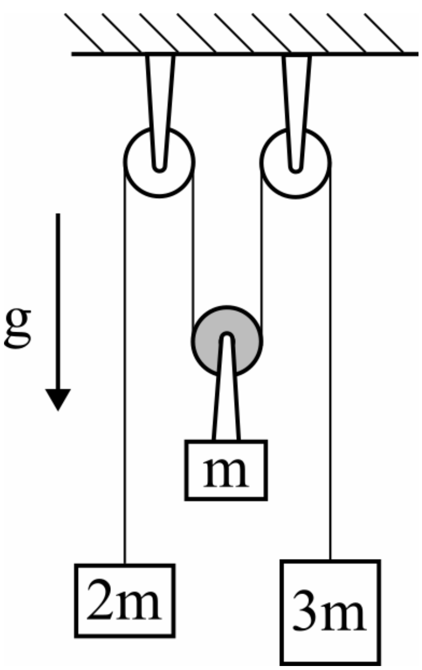
Lior Silberman's Math 428: Problem Set 2

Dynamics

For the next two problems we work in two dimensions with uniform gravity $-g$ in the vertical direction.

1. A hemisphere of radius r is fixed to a table. A particle of mass m moves on its (frictionless) surface, affected by constant vertical gravity $\underline{g}$. Place the coordinate system (x,y,z) where (x,y) are on the surface of the table and z is height off the table so that (for example) $\underline{g} = (0,0,-g)$.
   (a) Let $\underline{v}(t)$ be the velocity of the particle, $\underline{a}(t)$ its acceleration, and let $a_\perp$ be the component of the acceleration in the direction of the outward normal to the surface. Show that
$$a_\perp = -\frac{|\underline{v}|^2}{r}.$$
   (b) We place the particle at position $x(0) = (x_0, y_0, z_0)$ on the hemisphere and give it an initial velocity $\underline{v}(0)$ in some direction. At some point the particle will slide off the surface and become airborne; find the height $z(t)$
   (*c) *Where* is the particle when it comes off the surface?

2. Consider the system of masses and pulleys pictured in the figure. Assume that all pulleys and ropes are massless and frictionless. Recall that while you can't push on a rope, we assume that ropes don't stretch so the *tension* along a massless rope must be constant (the mass of any segment is zero).
   (a) Write down the equations of motion for the system; solve the equations.
   (b) Let x,y be the heights of the two hanging masses. Compute the position of the third mass as a function of x,y, and write down the total kinetic and potential energies of the system in terms of x,y and their time derivatives. Check that the total energy is conserved.
   (c) Suppose now that the mass m is actually the mass of the middle pulley, which therefore has moment of inertia $\frac{1}{2}mr^2$ (r is its radius). Again find the total kinetic and potential energies of the system.



3. Consider a system of N particles of masses m_i located at $x_i \in \mathbb{E}^d$. Let the force on the ith particle take the form $F_i + \sum_{j \neq i} F_{ij}$ where F_i is some external force while the interactions F_{ij} satisfy $F_{ij} = -F_{ji}$, Define the *centre of mass* of the configuration to be

$$x = \frac{1}{M} \sum_{i=1}^{N} m_i x_i \,,$$

where $M = \sum_i m_i$ is the total mass.

(a) Show that the centre of mass satisfies the equation of motion $M\ddot{x} = F$ where $F = \sum_i F_i$ is the total external force, and that the total momentum of the particles is $M\dot{x}$. Conclude that in the absence of external forces, the centre moves at constant velocity and total momentum is conserved.

(b) Two particles, initially moving at velocities v_1, v_2, collide with each other. Assume that the collection is purely elastic (this is a *mathematical* abstraction in which kinetic energy $\frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2$ is conserved), find the velocities after the collision.

Hint: make a Galilean transformation to the setup where the centre of mass is stationary.

The contraction mapping principle

A. Fix a complete metric space (X,d). Recall that this means that every Cauchy sequence in X converges.
   (a) Let $\{x_n\}_{n=0}^{\infty} \subset X$ be a sequence, and suppose that $\sum_{n=0}^{\infty} d(x_n, x_{n+1}) < \infty$. Show that the sequence is Cauchy (this holds in any metric space) hence converges.
   ? Let $f\colon X \to X$ be a Lipschitz continuous map with Lipschitz constant $\rho < 1$, in other words such that $d(f(x), f(y)) \leq \rho d(x,y)$. A *fixed point* of f is a point $x \in X$ such that $f(x) = x$.
   (b) Show that f has at most one fixed point.
   (c) Let $x_0 \in X$ be an arbitrary point, and define recursively $x_{n+1} = f(x_n)$. Show that problem (a) applies to this sequence.
   (d) Let $x_\infty = \lim_{n\to\infty} x_n$. Show that x_∞ is a fixed point for f.
   (e) Show that $d(x_0, x_\infty)$ is bounded in terms of the displacement $E_f(x_0) = d(x_0, f(x_0))$.

B. (Example: Newton's method) Let $F \in C^1(\mathbb{E}^n \to \mathbb{E}^n)$. Fixing $y \in \mathbb{E}^n$, given $x_n \in \mathbb{E}^n$ set $x_{n+1} = x_n - dF_{x_n}^{-1}(f(x_n) - y)$ if dF is invertible at x_n.
   (a) (Contraction) Suppose that there exists z with $F(z) = y$ and that dF_z is invertible. Show that for r small enough, the function $f(x) = x - dF_x^{-1}(F(x) - y)$ is a well-defined contraction on $B(z,r)$.
   (b) (Quadratic convergence) Suppose that F is twice differentiable at z. Setting $\varepsilon = |x - z|$ show that there is $C > 0$ so that if ε is small enough, then $|f(x) - z| \leq C\varepsilon^2$.

C. (Example: the implicit function theorem) Let $U \subset \mathbb{E}^n \times \mathbb{E}^m$ be open and let $F \in C^1(U; \mathbb{E}^m)$. Suppose that for some $(x_0, y_0) \in U$ the linear map $\frac{\partial F}{\partial y}|_{(x_0,y_0)} = dF_{(x_0,y_0)} \restriction_{\mathbb{R}^m} : \mathbb{R}^m \to \mathbb{R}^m$ has full rank. Then there exists a neighbourhood $x_0 \in V \subset \mathbb{E}^n$ and a function $g \in C^1(V; \mathbb{E}^m)$ such that $g(x_0) = y_0$, $\{(x, g(x))\}_{x \in V} \subset U$, and such that for $x \in V$ we have $F(x, g(x)) = F(x_0, y_0)$. If the neighbourhood V is small enough than any two such functions must agree on V.
   - Given x we want to solve the equation $F(x,y) = F(x_0, y_0)$ for y. For this let $A = dF_{(x_0,y_0)} \restriction_{\mathbb{R}^m}$ and given x set up the iterative scheme $G_x(y) = A^{-1}(F(x_0,y_0) - F(x,y))$.
   (a) There is a neighbourhood $(x_0, y_0) \in V \times Z \subset U$ such that if $x \in V$ then G_x is a contraction on Z.
   (b) Letting $g(x)$ be the fixed point of G_x. Show that $F(x, g(x)) = F(x_0, y_0)$.
   (c) Show that $g \in C^1(V; Z)$.
   (d) Suppose $F \in C^k(U; \mathbb{E}^m)$; show that $g \in C^k(U)$.

D. For a compact space K and a complete metric space (X,d) write $C(K;X)$ for the space of continuous functions $K \to X$.
   (a) Given $f, g \in C(X;K)$ show that the map $z \mapsto d_X(f(z), g(z))$ is continuous. Conclude that $D(f,g) \overset{\text{def}}{=} \sup\{d_X(f(z), g($ is well-defined.
   (b) Show that D is a metric on $C(X;K)$, the metric of *uniform convergence*.
   (c) Let $\{f_n\}_{n=0}^{\infty} \subset C(X;K)$ be a Cauchy sequence for the metric D. Show that for each $z \in Z$, $f_\infty(z) = \lim_{n\to\infty} f_n(z)$ exists.
   (d) Show that $\lim_{n\to\infty} D(f_n, f_\infty) \to 0$.
   (*e) Show that f_∞ is continuous, and conclude that $C(X;K)$ is complete.
   Hint ("3ε argument"): a Given $z_0 \in K$ and $\varepsilon > 0$ we need a neighbourhood U of z_0 such that $f_\infty(U) \subset B_X(z_0, \varepsilon)$. Observe that for each n, $f_n(z_0)$ can be made pretty close to $f_\infty(z_0)$, that if z is close enough to $z_0$then $f_n(z)$ is close to $f_n(z_0)$, and that $f_\infty(z)$ is close to $f_n(z)$.
   RMK We will use this result to study ODE.