Math 428: Problem Set 3, “direct method”

This one is mostly targeted at the graduate students.

Sobolev Embedding

For this group of exercises you may assume all functions are real-valued, valued in $\mathbb{R}^n$, or even valued in a Banach space (choose one for your solutions).

DEFINITION. Let $T\colon (V, \|\cdot\|_V) \to (W, \|\cdot\|_W)$ be a linear map between normed spaces. We say that T is *bounded* if there is a constant $C \geq 0$ such that for all $f \in V$, $\|Tf\|_W \leq C\|f\|_V$. We say that T is *compact* if for any $\{f_n\}_{n=1}^{\infty} \subset V$ such that $\|f_n\|_V$ are uniformly bounded, there is a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ such that $\{Tf_{n_k}\}_{k=1}^{\infty} \subset W$ is a Cauchy sequence with respect to $\|\cdot\|_W$.

1. (Banach space background)
   (a) Show that T is continuous in the norm topology iff it is bounded.
   (b) Show that any compact operator is bounded.
   (c) For a bounded operator T, its *operator norm* $\|T\| = \|T\|_{V\to W}$ is the infimum of all constants C as in the definition. Show that $\|T\| = \sup\left\{\frac{\|Tf\|_W}{\|f\|_V} : 0 \neq f \in V\right\}$ and that $\|\cdot\|_{V\to W}$ is a norm on the space $\mathrm{Hom}_{\mathrm{cts}}(V,W)$ of continuous linear maps.
   (d) Suppose that W is complete with respect to its norm. Show that $(\mathrm{Hom}_{\mathrm{cts}}(V,W), \|\cdot\|_{V\to W})$ is complete.
   (e) Suppose T is bounded. Show that it extends uniquely to a bounded linear map $\hat{T}\colon \hat{V} \to \hat{W}$ between the completions, and that $\left\|\hat{T}\right\|_{\hat{V}\to\hat{W}} = \|T\|_{V\to W}$.

DEFINITION. For a function f on the interval $[a,b]$, $k \in \mathbb{Z}_{\geq 0}$ and $0 < \alpha \leq 1$ set

$$\|f\|_{C^{\alpha}(a,b)} = \sup\left\{\frac{|f(t)-f(s)|}{|t-s|^{\alpha}} : a \leq s < t \leq b\right\};$$

$$\|f\|_{C^{k}(a,b)} = \sup\left\{\left|f^{(j)}(t)\right| : 0 \leq j \leq k, t \in [a,b]\right\};$$

$$\|f\|_{C^{k,\alpha}(a,b)} = \|f\|_{C^{k}(a,b)} + \left\|f^{(k)}\right\|_{C^{\alpha}(a,b)}.$$

For the first three notions we adopt the convention that if some derivative does not exist its value is ∞ and that if the supremum does not exist its value is ∞. Identifying $C^{k,0} = C^k$ we define for each k and $0 < \alpha \leq 1$

$$C^k(a,b) = \left\{f^{(k)} \text{ exists and is continuous on } [a,b]\right\}$$

$$C^{k,\alpha}(a,b) = \left\{f \in C^k(a,b) \mid \|f\|_{C^{k,\alpha}} < \infty\right\}.$$

2. (Spaces of continuous functions)
   (a) Show that $\|\cdot\|_{C^k(a,b)}$ is a norm on $C^k(a,b)$ (in particular, that it is finite for every function there) similarly for $C^{k,\alpha}$. Show that these spaces are complete.
   (b) Show that $\|\cdot\|_{C^k}$ is equivalent to the norm $\sup\{|f(t)|\} + \sup\left\{\left|f^{(k)}(t)\right|\right\}$.
   (c) Show that for $0 < \alpha < \beta < 1$ we have
   $$C^k(a,b) \supsetneq C^{k,\alpha}(a,b) \supsetneq C^{k,\beta}(a,b) \supsetneq C^{k,1}(a,b) \supsetneq C^{k+1}(a,b).$$
   (d) Show that the inclusions are bounded.
   (e) Show that these inclusions (other than $C^{k,1} \subset C^{k+1}$) are *compact*.
   Hint: Arzela–Ascoli.

3. For $f \in C^1(a,b)$ define $\|f\| = \int_a^b \left(|f(t)|^2 + |f'(t)|^2\right) dt$.
   (a) Show that $\|f\|_{H^1}$ is a norm on $C^1(a,b)$.
   (b) Show that $|f(t) - f(s)| \leq \|f\|_{H^1} \sqrt{|t-s|}$.
   (c) Find a constant K so that $\|f\|_{C^0} \leq K \|f\|_{H^1}$.
   (d) Let $H^1(a,b)$ be the completion of $C^1(a,b)$ with respect to this norm. Show that the identity inclusion $\iota\colon C^1(a,b) \hookrightarrow C(a,b)$ extends to a continuous embedding $H^1(a,b) \hookrightarrow C(a,b)$.

DEFINITION. Let f, g be reasonable functions on $[a,b]$ (say locally L^1). Say that g is a *weak derivative* of f if for $\phi \in C^\infty(\mathbb{R})$ with $\operatorname{supp}\phi \subset [a,b]$ we have
$$\int_a^b g(t)\phi(t)dt = -\int_a^b f(t)\phi'(t)dt.$$

4. Write f' for a weak derivative of f.
   (a) Show that if $f \in C^1$ then the usual derivative is a weak derivative.
   For the rest of the problem suppose that 0 is a weak derivative of f.
   (b) Show for all $\phi \in C_c^\infty(\mathbb{R})$ there is a constant k so that $\int_a^b f(t)\phi(t)dt = k\int_a^b \phi(t)dt$.
   (c) Show that $f = k$ almost everywhere.
   (d) Conclude that the weak derivative is unique: if g, g' are both weak derivatives then $g = g'$ a.e.

DEFINITION. For a a function f on $[a,b]$ such that its weak derivatives exist up to the kth order set
$$\|f\|^2_{H^k(a,b)} = \int_a^b \left(\sum_{j=0}^{k} \left|f^{(j)}(t)\right|^2\right) dt$$
(and set the norm to be infinite if the derivatives don't exist or if the weak derivatives are not square-integrable.

5. (Completeness)

(a) Show that $H^k(a,b)$ is complete, that $C^k \subsetneq H^k$, and that the inclusion is continuous with respect to the norms.
(b) Show that C^k is dense in H^k.
COR H^k can also be defined as the completion of C^k with respect to the H^k norm.

6. (Embedding) Let $k \geq 1$.
   (a) Show that the identity map $\left(C^k, \|\cdot\|_{H^k}\right) \to \left(C^{k-1,\frac{1}{2}}, \|\cdot\|_{C^{k-1,\alpha}}\right)$ is continuous.
   Hint: problem 4(b)
   (b) Conclude that there is a continuous inclusion $H^k \hookrightarrow C^{k-1,\frac{1}{2}}$ and a compact embedding $H^k \hookrightarrow C^{k-1}$.

DEFINITION. We say a function f is *locally* H^k, $C^{k,\alpha}$, etc if for every interval $[a,b]$ the restriction of f to that interval has this property.

7. ("continuity of wavefunctions") Fix a locally bounded function V and a real number E, and suppose $f \in L^2(\mathbb{R})$ satisfies $-\frac{\hbar^2}{2m}f'' + Vf = Ef$ in the sense of weak derivatives.
   (a) Show that $f \in H^{2,\text{loc}}(\mathbb{R})$.
   (b) Show that f and f' are continuous functions.

Existence of Minimizers in 1d

Fix an interval $I = [t_0, t_1]$ and let $M\colon \mathbb{R}^n \times I \to M_n(\mathbb{R})$ be continuous. Suppose that $M(q;t)$ is positive-definite and abounded below (as a quadratic form) by $\mu > 0$. Let $U\colon \mathbb{R}^n \times I \to \mathbb{R}$ be continuous and bounded above. Fixing $a, b \in \mathbb{R}^n$ we consider the variational problem of minimizing

$$S(\gamma) = \int_{t_0}^{t_1} \left[\langle \dot{\gamma}(t), \dot{\gamma}(t)\rangle_{M(\gamma(t);t)} - U\left(\gamma(t);t\right)\right] dt\,,$$

initially defined for $\gamma \in C^\infty(I;\mathbb{R}^n)$.

8. (interpretation)
   (a) Show that $\gamma \mapsto S(\gamma)$ is locally uniformly continuous in the H^1 norm, and hence extends to a continuous function $S\colon H^1(I;\mathbb{E}^n) \to \mathbb{R}$.
   (b) Show that S is bounded below on H^1.
   (c) Show that $D = \left\{\gamma \in H^1 \mid \gamma(t_0) = a,\, \gamma(t_1) = b\right\}$ is a closed subset (actually an affine subspace).

9. (Minimizers) Let $\{\gamma_n\}_{n=1}^\infty \subset D$ be a minimizing sequence, that is a sequence for which $S(\gamma_n) \to \inf_{\gamma \in D} S(\gamma)$.
   (a) Show that $\{\gamma_n\}_{n=1}^\infty$ are uniformly bounded in H^1 norm.
   (b) Show that after passing to a subsequence we may assume that γ_n converge weakly to $\gamma \in H^1$ and uniformly in $C^0(I)$.

(c) Does the uniform limit have to be γ?

(d) Show that $S(\gamma) \leq \liminf_n S(\gamma_n)$, and conclude that $S(\gamma) = \min_{\gamma \in D}$.
Hint: use the convexity of the kinetic term and the fact a weak limit of a sequence is the strong limit of a sequence of convex combinations of elements from the sequence.

**10. Supposing M and U are differentiable with respect to q, show that a minimizer in H^1 satisfies (in a weak sense) the Euler–Lagrange equation, and conclude that under such hypotheses it is in fact in C^1.