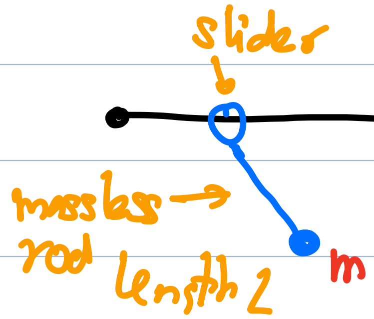
Math 428/603, lecture 2, 9/1/2025

Last time: Configuration space:
ambient $\mathbb{E}^d$, N particles have ambient $(\mathbb{E}^d)^N$.
add constraint $\underline{F}(x)=\underline{0}$ for $\underline{F}: \mathbb{E}^{dN} \to \mathbb{R}^m$.

Today: Coordinates on $X = \underline{F}^{-1}(\underline{0})$.
+ differentiation, velocity

Example:



slider free to slide left-right,
rod swing freely in plane of figure

Ambient space is $\mathbb{E}^2$, say slider at (x_s, y_s), mass at (x,y)

Constraint:

$$\begin{cases} x_s \in [a,b] \\ y_s = 0 \\ (x-x_s)^2 + (y-y_s)^2 = L^2 \end{cases}$$

$$\underline{F}(x_s, y_s, x, y) = \left(y_s, (x-x_s)^2 + (y-y_s)^2 - L^2\right)$$

"Instead use coords x_s, θ ← angle of rod with vertical
The slider is at $(x_s, 0)$
mass is at $(x_s + L\sin\theta, -L\cos\theta)$"

Also "The potential energy is $U = mgl\cos\theta$
The kinetic energy is
$$\frac{1}{2}m[\dot{x}^2+\dot{y}^2] = \frac{1}{2}m[\dot{x}_s^2 + 2l\cos\theta\,\dot{x}_s\dot{\theta} + l^2\dot{\theta}^2]"$$

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§3 Co-ordinates

First: derivatives. Let $\underline{F}: \mathbb{E}^n \to \mathbb{E}^m$.
We say $\underline{F}$ is differentiable at $x \in \mathbb{E}^n$ if there is a linear map $d\underline{F}_x : \mathbb{R}^n \to \mathbb{R}^m$ st.

$$\underline{F}(x+\underline{v}) = \underline{F}(x) + d\underline{F}_x(\underline{v}) + \underline{R}(x,\underline{v})$$

with $\dfrac{|\underline{R}(x,\underline{v})|}{|\underline{v}|} \to 0$ as $|\underline{v}| \to 0$.

in co-ords this is the Jacobian matrix

Observation: When $m=1$, dF_x is $\boxed{\text{not a vector}}$.
It is a linear map from $\mathbb{R}^n \to \mathbb{R}$, i.e. a linear functional := dual vector = covector.

If we wish to speak of the gradient vector $\vec{\nabla}F$ we need a way to identify dual space $(\mathbb{R}^n)^*$ with $\mathbb{R}^n$, ie. an inner pdt / metric. Choice of metric matters

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On configuration space, let x, x' be close:
write $x' = x + \varepsilon \underline{v}$ with $\underline{v}$ unit vector, ε small.
Then $\underline{0} = \underline{F}(x') = \underline{F}(x + \varepsilon \underline{v}) = \underset{\underline{0}}{\underline{F}(x)} + \varepsilon \, d\underline{F}_x(\underline{v}) + o(\varepsilon)$

$$\Rightarrow d\underline{F}_x(\underline{v}) = \frac{o(\varepsilon)}{\varepsilon} = o(1)$$ ← quantity → 0 as $\varepsilon \to 0$

As $x' \to x$, $\underline{v}$ will have convergent subsequences
$\Rightarrow$ in limit get vector $\underline{v}$ s.t. $d\underline{F}_x(\underline{v}) = 0$
$\Leftrightarrow \underline{v} \in \ker d\underline{F}_x$.

Def: The space tangent to X at x is

$$T_x X \overset{\text{def}}{=} \ker d\underline{F}_x.$$

$\mathbb{R}^{dN}$

ideally, $\dim T_x X = dN - m$

Notation: use Newton's dot to denote $\frac{d}{dt}$.
If $\gamma : I \to \mathbb{E}^{dN}$, $\dot{\gamma}(t) \in \mathbb{R}^{dN}$ is the usual derivative, also image of $\frac{d}{dt} \in T_t \mathbb{E}^1$ by $d\gamma_t$.

$$\partial_{\underline{v}} = \lim_{h \to 0} \frac{\underline{F}(x + h\underline{v}) - \underline{F}(x)}{h}$$

Ex: $\gamma : I \to \mathbb{E}^{dN}$ is diff, suppose $\gamma(t_0) \in X$.
Then $\forall t \; \gamma(t) \in X$ iff $\forall t \; \dot{\gamma}(t) \in T_{\gamma(t)} X$

Constraint on motion: $\dot{\gamma}$ restricted to subspace of $T_x X$, usually given by linear functional $T_x X \to \mathbb{R}$

Coordinates

Theorem: (Implicit function thm) let $x_0 \in \Omega \subset \mathbb{E}^n$, $\underline{F}: \mathbb{E}^n \to \mathbb{E}^m$, ctsly diff, rk $d\underline{F}_{x_0} = m$.

Then can choose m co-ords of $\mathbb{E}^n$ so that locally in $\mathbb{E}^n$, on $\Sigma = \underline{F}^{-1}(\underline{F}(x_0))$, these co-ords are functions of other ones.

Examples $F(x,y) = x^2+y^2-1$ on $\mathbb{R}^2$

$dF_{(x,y)} = 2x\,dx + 2y\,dy$

on $\Sigma = \{x^2+y^2=1\}$ $\mathrm{rk}_{\mathbb{R}}\, dF_{(x,y)} = 1$

if $y \neq 0$ locally make y fcn of x: $y = \pm\sqrt{1-x^2}$

if $x \neq 0$ " " x " " y: $x = \pm\sqrt{1-y^2}$

Also, function so defined has expected derivative, if $\underline{F}$ is k-times diff so is that function

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Cor: (Inverse fcn thm) If $n=m$ and rk $d\underline{F}_x = n = m$ the F is invertible (locally)

Def. A co-ordinate is any function (U, q) $U \subset X$ open, $q : U \to \mathbb{R}$ is a function

A system of co-ordinates is a tuple $q = (U, \{q_\alpha\}_{\alpha=1}^{\dim X})$ where $q_\alpha : U \to \mathbb{R}$, dq is invertible on U

dq? If $x, x' \in X$ close, $x' = x + \underline{v} + \underline{e}$ with $\underline{v} \in T_x X$, $\underline{e} = o(|\underline{v}|)$

so if $\underline{f} : X \to \mathbb{R}^r$, say $\underline{f}$ is diff if
$$\underline{f}(x') = \underline{f}(x) + d\underline{f}_x \, \underline{v} + o(|x - x'|)$$
for $d\underline{f}_x : T_x X \to \mathbb{R}^r$.

On $\vec{F} = m\vec{a}$: $\langle \cdot, \cdot \rangle : V \to V^*$
given $\underline{v}$ set $\underline{v}^*(\underline{w}) = \langle \underline{v}, \underline{w} \rangle$
and Riesz Representation Theorem:
this is an isom $V \to V^*$ (if $\dim V < \infty$)