Math 428, lecture 3, 9/1/2025

Last week: Config. space: collect states of a mechanical system in $\mathbb{E}^d$ as a subset $\mathbb{X} \subseteq (\mathbb{E}^d)^N$ cut out by holonomic constraint

$\Rightarrow$ tangent space $T_x\mathbb{X}$ where velocities live

This week: Newtonian mechanics

Moving from kinematics to dynamics, which is to say predicting future from present. It turns out that the "present" is $(x, \underline{v})$ where $x \in \mathbb{X}$, $\underline{v} \in T_x\mathbb{X}$

$\Rightarrow$ get a dynamical system with state space
$$T\mathbb{X} = \{(x, \underline{v}) : x \in \mathbb{X},\ \underline{v} \in T_x\mathbb{X}\}$$

§1. Newton's Laws

Equip vector space $\mathbb{R}^d$ with std inner product, distance $|\underline{v} - \underline{v}'| = \sqrt{\sum_i (v_i - v_i')^2}$.

Euclidean space $\mathbb{E}^d$ is the affine space modelled on $\mathbb{R}^d$, i.e. a principal homogeneous space for $\mathbb{R}^d$. We write $x+\underline{v}$ for the translate of $x \in \mathbb{E}^d$ by $\underline{v} \in \mathbb{R}^d$, set distance

$$d(x,x') = |x'-x| \qquad (\text{"Euclidean distance"})$$

If $\gamma : I \to \mathbb{E}^d$ is a curve, $\gamma(t+h) - \gamma(t) \in \mathbb{R}^d$ is a displacement, can divide by h, if $\gamma \in C^1$ we can talk about velocity $\dot{\gamma}(t) \in \mathbb{R}^d$.

In co-ords if $\gamma(t) = (x_1(t), .., x_d(t))$

$$\dot{\gamma}(t) = (\dot{x}_1(t), .., \dot{x}_d(t))$$

Clear extension to N particles

Axioms (Newton's 2nd Law) There is a function $F : \mathbb{E}^{dN} \times \mathbb{R}^{dN} \times \mathbb{R} \to \mathbb{R}^{dN}$ (called "force") so that the path of the system satisfies

$$M\ddot{x} = F(x, \dot{x}; t).$$

For the moment M is a diagonal matrix

multiplying the co-ords assoc to jth particle by number $m_j > 0$

should take
(in fact $m_j : \mathbb{R}^d \to (\mathbb{R}^d)^*$ be the inner pdt rescaled by constant m_j, and F is in $(\mathbb{R}^{dN})^*$)

Remark: F represents the sum of (1) interactions (2) external forces (3) constraint forces

Remark: Write the system as

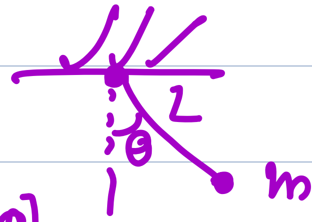
$$\frac{d}{dt}\begin{pmatrix} x \\ \dot{x} \end{pmatrix} = \begin{pmatrix} \dot{x} \\ m^{-1} F(x, \dot{x}; t) \end{pmatrix}$$

we see that it suffices to analyse DE of the form $\dot{y} = f(y; t)$ for $y \in \mathbb{R}^n$. Also that the "state of motion" is the pair $\begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix}$.

Examples (0) The free particle has $m\ddot{x} = 0$ ("Newton's 1st law": solution is $x(t) = x(0) + \underline{v}\, t$)

(1) Hookean spring $m\ddot{x} = -kx$ ("Harmonic oscillator")

(2) The physical pendulum



m is at $(x,y) = (L\sin\theta, -L\cos\theta)$

Subject to two forces:

(1) gravity: constant $F = (0, -mg)$

(2) attachment to rod T.

decomposing F in direction of T and perpendicular get $|T| = mg\cos\theta$

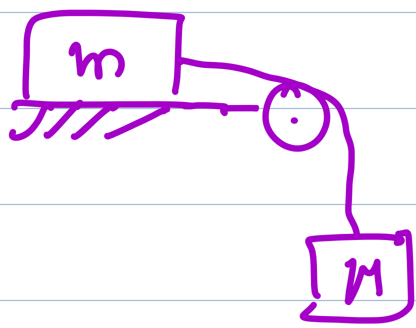
$$\Rightarrow \quad m\ddot{x} = mg\cos\theta\sin\theta$$

$$\vdots$$

$$\Rightarrow \quad \boxed{mL\ddot{\theta} = -mg\sin\theta}$$

(if θ is small, approximate by $\ddot{\theta} = -\frac{g}{L}\theta$)

(3)



"rope and pulley system": typically assume rope massless

Mass m incur a force due to gravity, a constraint force ("Normal force"), the attachment T to rope.

Mass M incurs T, gravity.

§2. Existence & Uniqueness for ODE

Physics: the physical system always does the same thing if started from same (x_0, v_0).

Maths: Should be a theorem to that effect
 – this is a test of the model

Defn: An ordinary differential equation is a pair (Ω, F) where $\Omega \subset \mathbb{E}^n \times \mathbb{E}^1$ is open and $F: \Omega \to \mathbb{R}^n$ is cts

A solution to the ODE is a pair (I, γ) where $I \in \mathbb{E}^1$ is an interval, $\gamma \in C^1(I, \mathbb{E}^n)$ s.t.
(1) for all $t \in I$, $(\gamma(t), t) \in \Omega$
(2) – " – $\dot{\gamma}(t) = F(\gamma(t), t)$

Def: Say F is locally Lipschitz if for every $(y_0, t_0) \in \Omega$ there is a neighbourhood $(y_0, t_0) \in U \subset \Omega$, and a number L s.t.
If
$$(y,t), (y',t) \in U \text{ then}$$
$$|F(y,t) - F(y',t)| \leq L \cdot |y-y'|$$

Examples If $F \in C^1$ this holds by MVT.

Prop: (Picard iteration) Suppose F is locally Lipschitz. Then for all $(y_0, t_0) \in \Omega$, there is $\varepsilon > 0$ so that the equation has a unique solution on $I = (t_0 - \varepsilon, t_0 + \varepsilon)$ with $\gamma(t_0) = y_0$

Lemma: There are $\varepsilon, R, L, M > 0$ st.

(1) $B = B(y_0, R) \times [t_0 - \varepsilon, t_0 + \varepsilon] \subset \Omega$
(2) on B, $|F| \leq M$
(3) For all $(y,t), (y',t) \in B$, $|F(y,t) - F(y',t)| \leq L|y-y'|$
(4) We have $\varepsilon \leq \frac{R}{M+1}$, $\varepsilon \leq \frac{1}{L+1}$

Pf: Let U be a nbd of (y_0, t_0) on which we have Lipschitz constant L.

If $R, \tilde{\varepsilon}$ small enough, $\tilde{B} = B(y_0, R) \times [t_0 - \tilde{\varepsilon}, t_0 + \tilde{\varepsilon}] \subset U$

Since $\tilde{B}$ is cpt, F cts, F is bdd on $\tilde{B}$, say $|F| \leq M$ on $\tilde{B}$.

Let $\varepsilon = \min\left\{\frac{R}{M+1}, \frac{1}{L+1}, \tilde{\varepsilon}\right\}$.

Lemma: Let $I = (t_0 - \varepsilon, t_0 + \varepsilon)$, let $\gamma \in C^1(I, \mathbb{E}^n)$ be a solution on I, $\gamma(t_0) = y_0$. Then $\gamma(t) \in B(y_0, R)$ for all t.

Pf: Informally, let t be the first time we hit $\partial B(y_0, R)$ until time t, on $[t_0, t]$ we are in B, so on (t_0, t), $|\dot{\gamma}(s)| = |F(\gamma(s), s)| \leq M$

so $|\gamma(t) - \gamma(t_0)| \leq M|t - t_0| \leq M\varepsilon < R$.

Proof of Prop: Let $I = [t_0 - \varepsilon, t_0 + \varepsilon]$

let $C(I, \mathbb{E}^n)$ is the space of cts functions $I \to \mathbb{E}^n$, equipped with distance

$$d(\gamma, \gamma') = \sup_{t \in I} |\gamma(t) - \gamma(t')|$$

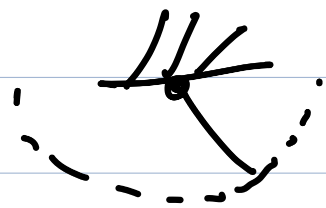
$$= \|\gamma - \gamma'\|_{L^\infty(I)}$$

let $X = \{\gamma \in C(I; \mathbb{E}^n) \mid \gamma(t) \in B(y_0, R)\ \forall t\}$

Idea: define given $\gamma \in X$

$$G(\gamma)(t) = y_0 + \int_{t_0}^{t} F(\gamma(s), s)\, ds$$

Example on pendulum



X = semicircle in plane

Define $U : X \to \mathbb{R}$ by $U(x) = -mgl\cos(\theta(x))$

= composition of $V(\alpha) = -mgL\cos\alpha$

and of $\Theta(x)$

$u = v \circ \Theta$

$v = u \circ (\Theta^{-1})$

Θ^{-1} is the function $\alpha \mapsto (L\sin\alpha, -L\cos\alpha)$