Math 428, Lecture 7, 16/1/2025

Last time: Dynamics of particles moving in $\mathbb{E}^d$.

Newton's 2^{nd} law : $x \in \mathbb{E}^{dN}$ satisfies

$$M\ddot{x} = F(x, \dot{x}; t)$$

Called F "force".

$\Rightarrow$ "state of motion" $\begin{pmatrix} x \\ \dot{x} \end{pmatrix}$ satisfies $\frac{d}{dt}\begin{pmatrix} x \\ \dot{x} \end{pmatrix} = \begin{pmatrix} \dot{x} \\ M^{-1}F \end{pmatrix}$

Prop: Let $\dot{y} = F(y; t)$ be an ODE defined on $\Omega \subset \mathbb{E}^n \times \mathbb{E}^1$. Suppose F is cts on Ω, locally Lipschitz in 1^{st} var.

For every initial condition (y_0, t_0) there is an $\varepsilon > 0$ and a unique solution $y(t)$ of the ODE : $(y(t), t) \in \Omega$ for all $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ and $F(y(t), t) = \dot{y}$.

Lemma: Have $\varepsilon, R, L, M > 0$ s.t. :

(1) $B = B(y_0, R) \times [t_0 - \varepsilon, t_0 + \varepsilon] \subset \Omega$

(2) For all $(y, t) \in B$, $|F(y, t)| \leq M$

(3) For all $(y, t), (y', t) \in B$, $|F(y, t) - F(y', t)| \leq L|y - y'|$

(4) $\varepsilon \leq \frac{R}{M+1}$; $\varepsilon \leq \frac{1}{L+1}$

Lemma: Let $\gamma \in C^1(I; \mathbb{E}^n)$, $I = (t_0 - \varepsilon, t_0 + \varepsilon)$. Suppose γ is a solution to ODE with $\gamma(t_0) = y_0$. Then $\gamma(t) \in B(y_0, R)$ for all $t \in I$.

Proof of Prop: ("Picard iteration")

In $C(I; \mathbb{E}^n)$ let $X = \{\gamma : \gamma(t) \in B(y_0, R)$ for all $t \in I\}$.

Equipped with $d(\gamma, \gamma') = \|\gamma - \gamma'\|_{L^\infty(I)} = \sup_{t \in I} |\gamma(t) - \gamma'(t)|$

(recall $C(I, \mathbb{E}^n)$, thus X, is complete wrt this metric)

Given $\gamma \in X$ set $(G(\gamma))(t) = y_0 + \int_{t_0}^{t} F(\gamma(s); s)\,ds$.

Note that for $|s - t_0| \leq \varepsilon$, always have $(\gamma(s), s) \in B$

so $|F(\gamma(s), s)| \leq M$, so $|(G(\gamma))(t) - y_0| \leq M\varepsilon < R$

so $G(\gamma)(t) \in B(y_0, R)$.

Also by FTC, $G(\gamma)$ is cts, so $G(\gamma) \in X$.

Clearly γ is a solution iff $G(\gamma) = \gamma$

Key: If $\gamma, \gamma' \in X$, $t \in I$,

$$|G(\gamma)(t) - G(\gamma')(t)| \leq \left| \int_{t_0}^{t} (F(\gamma(s); s) - F(\gamma'(s), s))\,ds \right|$$

$$\leq \int_{t_0}^{t} |F(\gamma(s); s) - F(\gamma'(s); s)|\,ds$$

$$\leq L \int_{t_0}^{t} |\gamma(s) - \gamma'(s)| ds \leq L \, \varepsilon \, d(\gamma, \gamma')$$

Take sup over t, get $d(G(\gamma), G(\gamma')) \leq \frac{L}{L+1} d(\gamma, \gamma')$

By contractive mapping principle, G has a unique fixed point. ∎

Theorem: (Existence & Uniqueness Theorem)

Let (Ω, F) be a Lipschitz ODE, $(y_0, t_0) \in \Omega$

(1) There exists a local solution $\gamma : I \to \mathbb{E}^n$, for some $I = (t_0 - \varepsilon, t_0 + \varepsilon)$

(2) (Uniqueness) if (I, γ), (I', γ') are both solutions, then $\gamma = \gamma'$ on $I \cap I'$.

(3) (Blowup) There is a maximal solution (I_{max}, γ_{max}) s.t. every solution is $\gamma_{max}|_I$, $t_0 \in I \subset I_{max}$.

If $I_{max} = (a, b)$ and a or b is finite then $\gamma(t)$ "escapes" at $t \to a^+$ or $t \to b^-$ respectively in the sense that for all $K \subset \Omega$ cpt, if t is close enough to endpoint $(\gamma(t), t) \notin K$.

(4) If $F(\gamma(t), t) = F_0(\gamma(t))$ ("automous") then must escape compact sets of $\mathbb{E}^n$.

Today: Galilean symmetry

Divide forces into "interactions": F_{ij} = force jth particle exerts on ith

"external": F_i

s.t. $F(x,\dot{x};t)_i = F_i(x_i,\dot{x}_i;t) + \sum_{j\neq i} F_{ij}(x_i,x_j,\dot{x}_i,\dot{x}_j,t)$

Assume $F_{ij} = F_{ij}(x_i,x_j)$

Extend affine space $\mathbb{E}^d$ to spacetime $A^{d,1} = \mathbb{E}^d \times \mathbb{E}^1$

points $(x,t)\in A^{d,1}$ called "events".

call $t: A^{d,1}\to \mathbb{E}^1$ "time".

Subset $\mathbb{E}^d\times\{t\}$ is a timeslice.

Ex: $\mathrm{Isom}(\mathbb{E}^d)$ contains $\mathbb{R}^d$ acting by translation, point stabilizer $= O(d)$ $\Rightarrow$ $\mathrm{Isom}(\mathbb{E}^d) \cong O(d)\ltimes\mathbb{R}^d$
$\subseteq \mathrm{Aff}(\mathbb{E}^d)$

Def: The Galilean group are the affine maps $g\in \mathrm{Aff}(A^{d,1})$ s.t.

(1) preserve slices setwise

(2) preserve distance on slices

Ex: (1) Any rigid motion of $\mathbb{E}^d$.

(2) 'boosts': $(x,t)\to(x+\underline{u}t,t)$ $\quad \underline{u}\in\mathbb{R}^d$ fixed

Axiom: The laws of physics are invariant under Galilean symmetry.

Lemma: Say $F_{ij} = F_{ij}(x_i, x_j)$. Then $|F_{ij}|$ can only depend on $|x_i - x_j|$, direction of F_{ij} along $x_i - x_j$.

Pf: For any $(x_1, x_2) \in \mathbb{E}^{d \cdot 2}$, (x_1', x_2') if $|x_1 - x_2| = |x_1' - x_2'|$

have rigid motion $g \in \mathrm{Isom}(\mathbb{E}^d)$ st. $g(x_1) = x_1'$, $g(x_2) = x_2'$

Also, any rotation in $\underset{\mathbb{R}^{d-1}}{\underset{\cong}{(x_i - x_j)^\perp}}$ will fix x_i, x_j

more component of F_{ij} perpendicular to $x_i - x_j$.

Example: Surface $\{x^2+y^2+z^2=1\} = S^2$ On notion of "d_q" in notes

near $(0,0,1)$ can have $z = z(x,y)$ $\left(z = \sqrt{1-x^2-y^2}\right)$

claim: if (x,y) close to $(0,0)$

then $(x,y,z(x,y)) - (0,0,1) = (x,y,0) + O(|x|^2+|y|^2)$

say $z(x,y) = z_0 + z_1(x,y) + o(|x|+|y|)$

linear

Tangent vector in direction (x,y): $(x, y, z_1(x,y))$

$$T_{(0,0,1)} S^2 = \cancel{\ker}\ \ker\ df_{(0,0,1)}$$

$f(x,y,z) = x^2+y^2+z^2$

curve $(x(t), y(t)) \rightsquigarrow$ curve $\gamma: (x(t), y(t), z(x(t),y(t)))$

on surface $\dot{\gamma}(0) \in T_x X$

$$f(x, y, z(x,y)) = 1$$

$$\Rightarrow df \circ (\mathrm{Id}, dz) = 0 \qquad \Rightarrow df(\text{result}) = 0$$

identity

Time-dependent co-ords

$U \subset \mathbb{X} \times \mathbb{E}^1$ time axis

$q : U \to \mathbb{R}^{\dim \mathbb{X}}$

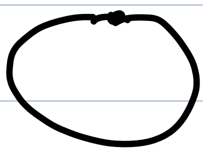
$q \in C^1$

$dq\big|_{T_x\mathbb{X}}$ invertible for each t

$T_{(x,t)}(\mathbb{X} \times \mathbb{E}^1) = T_x\mathbb{X} \oplus \overset{\mathbb{R}}{\overset{\downarrow}{T_t\mathbb{E}^1}}$

$\Rightarrow$ locally, get map $q^{-1}(\underset{\theta}{\cdot}, t)$ inverse to $q(\underset{x}{\cdot}, t)$

Example:



$(x,y) = (r\cos\theta, r\sin\theta)$

$\theta(x,y) = \arctan y/x$

$$\theta(x,y,t) = \arctan(y/x) - \omega t$$

$$(x,y) = (r\cos(\theta + \omega t), r\sin(\theta + \omega t))$$

$$\frac{dx}{dt} = -r\sin(\theta + \omega t)\left[\frac{d\theta}{dt} + \omega\right]$$

$$\frac{dy}{dt} = r\cos(\theta + \omega t)\left[\dot{\theta} + \omega\right]$$