Math 928, Lecture 5, 21/1/2025

PS2 in development

Last time: Galilean transformations

Today: (1) Example
(2) Energy & Work
(3) Conservation of momentum

If A, $\check{A}$ are $(d+1)$-dim space-times, a Galilean transformation is an affine bijection $f: A \to \check{A}$ s.t.:
(1) f preserves simultaneity
(2) on each timeslice f is an isometry.

Ex: Any Galilean map $A \mathcal{S}$ is a combination of translations (space & time), rigid motions, boosts. + semidirect prod structure

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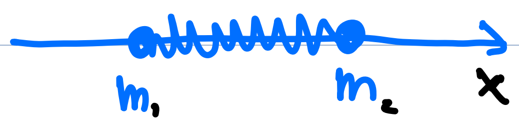
Example: $d=1$



Two masses connected by spring of rest length L.

say m_1 at x_1, m_2 at x_2

Force on m_1: $k(x_2 - x_1 - L)$ $\Rightarrow$ $\begin{cases} m_1 \ddot{x}_1 = k(x_2 - x_1 - L) \\ m_2 \ddot{x}_2 = k(x_1 - x_2 + L) \end{cases}$

Force on m_2: $k(x_1 - x_2 + L)$

notes $F_{12} = -F_{21}$ ("Newton's 3rd Law")

$\Rightarrow \quad m_1 \ddot{x}_1 + m_2 \ddot{x}_2 = 0 \quad \Rightarrow \quad \frac{d}{dt}\left(m_1 \dot{x}_1 + m_2 \dot{x}_2\right) = 0$

conserved quantity

total (linear) momentum

$M = m_1 + m_2$

Define $\quad u = \dfrac{m_1 \dot{x}_1 + m_2 \dot{x}_2}{M}$

Also have $\quad \ddot{x}_1 - \ddot{x}_2 = \frac{k}{m_1}(x_2 - x_1 - L) - \frac{k}{m_2}(x_1 - x_2 + L)$

set $\hat{y} = x_1 - x_2$ set $\ddot{\hat{y}} = -k\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\hat{y} - k\left(\frac{1}{m_1} + \frac{1}{m_2}\right)L$

set $\quad y = x_1 - x_2 + L$

then $\quad \ddot{y} = -k\left(\frac{1}{m_1} + \frac{1}{m_2}\right) y$

set $\quad \mu = \dfrac{1}{\frac{1}{m_1} + \frac{1}{m_2}}$, get $\mu \ddot{y} = -ky$

$\Rightarrow \quad y = A\cos(\omega t + \phi) \quad \omega = \sqrt{k\left(\frac{1}{m_1} + \frac{1}{m_2}\right)} = \sqrt{k/\mu}$

$\Downarrow$

$\dot{x}_1 - \dot{x}_2 = \dot{y}$, know $m_1 \dot{x}_1 + m_2 \dot{x}_2 = Mu$.

Better to work in co-ords $x = \dfrac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$, $y = x_1 - x_2 + L$

Different pov: change co-ords to $\hat{x}_1 = x_1 - ut$, $\hat{x}_2 = x_2 - ut$

$\ddot{\hat{x}}_1 = \ddot{x}_1 = k(x_2 - x_1 - L) = k(\hat{x}_2 - \hat{x}_1 - L)$

same for $\hat{x}_2$.

in both spacetimes, the Hookean spring law describes force same way.

Energy & Work

Have N particles moving on $\mathbb{E}^d$. If at time t, jth particle is at x_j it has velocity $\dot{x}_j \in T_{x_j}\mathbb{E}^d \cong \mathbb{R}^d$

Euclidean structure: on $\mathbb{R}^d$ have inner product, a linear map $g: \mathbb{R}^d \to (\mathbb{R}^d)^*$ (s.t. $g^* = g$, $\langle gv, v \rangle \geq 0$)

$(gw)(v) = \langle w, v \rangle$

Notation: if $v, w \in \mathbb{R}^d$, $\langle w, v \rangle$ = inner pdt
if $v \in \mathbb{R}^d$, $w^* \in (\mathbb{R}^d)^*$, $\langle w^*, v \rangle$ = pairing

Say jth particle has mass m_j. Get map M:

$$M: \mathbb{R}^{dN} \to (\mathbb{R}^{dN})^*$$

$$M = \bigoplus_j m_j g_j \ : T_x\mathbb{E}^{dN} \to T_x^*\mathbb{E}^{dN}$$

in orthogonal co-ords, g is matrix $\begin{pmatrix} 1 & & \\ & 1 & \\ & & \ddots \end{pmatrix}$

$$M = \begin{pmatrix} m_1 I_d & & & \\ & m_2 I_d & & \\ & & \ddots & \\ & & & m_N I_d \end{pmatrix}$$

Def: The kinetic energy of the system (at time t while moving along $x(t)$) is:

$v(t) = \dot{x}(t)$ $\qquad$ $v_j(t) = \dot{x}_j(t)$

$$T = \frac{1}{2}\langle M\underline{v}, \underline{v} \rangle = \frac{1}{2}\sum_j m_j \langle g\underline{v}_j, \underline{v}_j \rangle = \frac{1}{2}\sum_j m_j |v_j|^2$$

Side discussion about ODE $\ddot{x} = F(x)$

mult by $\dot{x}$. Get $\underbrace{\ddot{x}\dot{x}}_{\frac{d}{dt}\left(\frac{1}{2}\dot{x}^2\right)} = F(x)\dot{x}$

If can compute $\int F(x)\dot{x}\,dt$, get one integration

E.g.: if $F(x) = dU$ for some $U = U(x)$

then $F(x)\dot{x} = \frac{d}{dt}U(x)$

Here, $\frac{dT}{dt} = \sum_j m_j \langle \ddot{q}_j, v_j \rangle = \sum_j \langle F_j, \underline{v}_j \rangle$

$= \langle F, \underline{v} \rangle$

$\Rightarrow$ Force is a dual vector.

Def: The work done by the Force is

$$\int \langle F, d\gamma \rangle = \int_{t_0}^{t_1} \langle F, \underline{v}(t) \rangle\,dt$$

$\Rightarrow$ ("conservation of energy") $T(t_1) - T(t_0) =$ work done.

Def: Call Force F conservative if $F = -dU$ for some $U = U(x)$ (locally)

Lemma: F conservative iff $\oint F\,d\gamma = 0$ for closed loops ("small" loops)

In this context call U the potential.

Observation: Constraint force = component of acceleration $\perp$ to constraint

does no work: $F \perp v$

Conclusion: Let F_j be the force on jth particle other than conservative & constraint forces.

~~Then~~ Let U be the potential for conservative forces. Then if $E = T + U$

$$\frac{dE}{dt} = \sum_j \langle F_j, v_j \rangle$$

check: If F_{ij} conservative, $F_{ij} = F_{ij}(x_i - x_j)$

$F_{ij} = -d_i U(x_i - x_j)$, $F_{ji} = -d_j U(x_i - x_j)$