Math 428, lecture 6, 23/1/2025

So far: (1) Newtonian mechanics:

(1) $F = ma$
(2) $\Rightarrow$ ODE
(3) change of coord by Galilean maps
(4) Kinetic energy, potential energy
$\Rightarrow$ conservation of energy

(2) Kinematics:

(1) constraints & configuration space
(2) differention + tangent space
(3) co-ordinates, parametrization

Today: Chapter 3: Lagrangian Mechanics

Historical summary: (1) Euler: equations of motion $F=ma$ can be written in a form that is indep of choice of co-ordinates

(2) Lagrange: same works under constraints
$\Rightarrow$ Euler-Lagrange equations

(3) Hamilton: can obtain the Euler-Lagrange equations from a variational problem.

§2. Calculus of Variations

Problem: Fix a bounded domain $\Omega \subset \mathbb{R}^r$ (with reasonable boundary $\partial\Omega$) consider the problem of minimizing

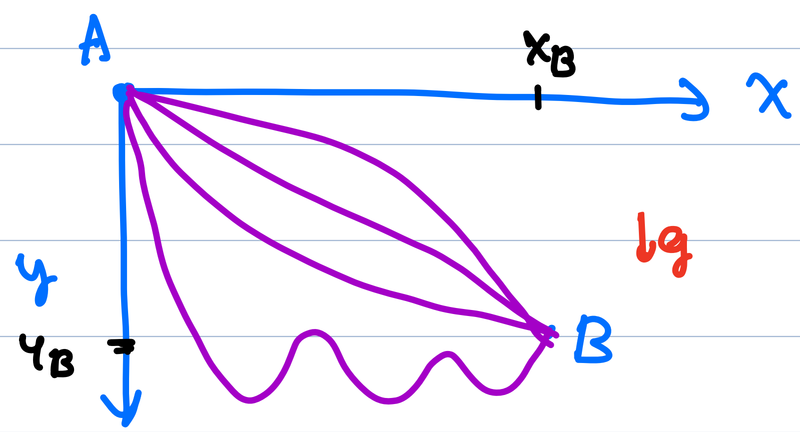
$$S = \int_\Omega L(u(t), du(t); t)\, dt$$

over the space of sufficiently nice $u : \Omega \to \mathbb{R}^n$, subject to $u\restriction_{\partial\Omega} = g$ (fixed). $g : \partial\Omega \to \mathbb{R}^n$

$$L : \mathbb{R}^n \times M_{n\times r}(\mathbb{R}) \times \Omega \to \mathbb{R} \qquad \text{sufficiently nice}$$

(for us $r=1$, $\Omega = (t_0, t_1)$, $\partial\Omega = \{t_0, t_1\}$, constraint $u(t_0) = a$, $u(t_1) = b$)

Example: (Brachistochrone) (Johann Bernoulli 1696 following Galileo)



problem: find curve $y = u(x)$ s.t. time to reach B from A is minimal (uniform gravity)

by conservation of energy, at $(x, u(x))$, $\frac{1}{2}mv^2 = mgu$

So $|v(x)| = \sqrt{2g u(x)}$. The length of curve from x to $x+dx$ is $\sqrt{1+u'^2}\, dx$, so the time to cover this part is

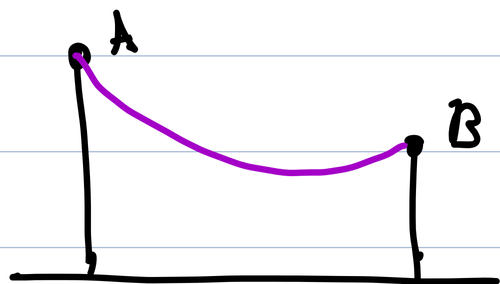
$$\frac{\sqrt{1+(u')^2}\, dx}{\sqrt{2g u(x)}}.$$

So total time is: $T = \int_0^{x_B} \sqrt{\frac{1+u'(x)^2}{2g u(x)}}\, dx$

Let $L(y, y') = \sqrt{\frac{1+(y')^2}{y}}$ defined on $\mathbb{R}_{>0} \times \mathbb{R}$

Then $T(u) = \int_0^{x_B} L(u(x), u'(x))\, dx$

want $\arg\min T(u)$ with $u(0)=0$, $u(x_B) = y_B$



__Example:__ (Catenary)

Fixed length of chain to be hung from A, B. Known: ("virtual work") total potential energy of chain is minimized

(length will be a constraint $\Rightarrow$ Lagrange multiplier)

__Example__ (Minimal surface): $\Omega \subseteq \mathbb{R}^r$ function u parametrizes hyper surface $(t, u(t)) \subset \mathbb{R}^{r+1}$

Area is $\int_\Omega \sqrt{1+|du|^2}\, dt$

Paradigm: to minimize $S(u)$, diff S wrt u, find critical points

Technicalities: (1) is there a minimum?
(2) what is "derivative"?

indirect approach: find critical pts, argue why they are minimal

direct approach: prove minimizer exists ⇒ must solve equation.

Remarks on differentiation

let V be some vector space (eg. $\{\gamma : I \to \mathbb{R}^n\}$ + diff conditions)
let $f : V \to \mathbb{R}$ be a "functional".

Def: Say f is

(1) Fréchét diff at $x \in V$ if (V Banach space)
$\exists$ linear functional $\lambda = df_x \in V'$ s.t.

differential

$$f(x+v) = f(x) + \langle \lambda, v \rangle + o(\|v\|)$$

(2) Gateaux differentiable at x if (V topological)
have $\lambda = df_x \in V'$ s.t.

directional derivative.

$$df_x(v) = \lim_{h \to 0} \frac{f(x+hv) - f(x)}{h} \quad \text{for all } v$$

For minimality, enough to look at $\lim_{h \to 0} \frac{f(x+hv)-f(x)}{h}$

non-uniformly in v, non-linearly, no need to take all v, only some.

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§4. Formal calculation

$I = [t_0, t_1]$, $\Omega \subset \mathbb{R}^n$ "co-ord patch

$\gamma : I \to \Omega$ path, $L : T\Omega \times I \to \mathbb{R}$, as nice as needed

$L = L(q, v; t)$

Get $S(\gamma) = \int_{t_0}^{t_1} L(\gamma(t), \dot{\gamma}(t); t)\, dt$

More on linear algebra

Given inner pdt $\langle \cdot, \cdot \rangle$ on V,

want to identify vector $w \leftrightarrow$ functional $\langle w, \cdot \rangle$

(Say want to minimize $F(x)$

Have guess x_0, compute dF_{x_0}, use inner pdt

to represent it by a vector ∇F_{x_0}

go to $x_1 = x_0 + h \cdot \nabla F_{x_0}$)

$$dF_{x_0}(v) = \langle \nabla F_{x_0}, v \rangle$$

pairing $V^* \times V$ inner pdt

if V_j vsp can make space $V = \bigoplus_{j=1}^{N} V_j$

$\underline{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix}$, $v_j \in V_j$

$$\langle \underbrace{\underline{v}}_{\text{vector}}, \underbrace{\underline{w}}_{\text{vector}} \rangle_V \overset{\text{def}}{=} \sum_{j=1}^{N} \langle v_j, w_j \rangle_{V_j}$$

if $g_j : V_j \to V_j'$ can make g by:

$$\langle \underbrace{g\begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix}}_{\text{functional}}, \underbrace{\begin{pmatrix} w_1 \\ \vdots \\ w_N \end{pmatrix}}_{\text{vector}} \rangle \overset{\text{def}}{=} \sum_{j=1}^{N} \langle g\, v_j, w_j \rangle$$

<u>Pendulum:</u> $x = R\sin\theta$, $y = -R\cos\theta$

$$\dot{x} = (R\cos\theta)\,\dot{\theta}\ , \quad \dot{y} = (R\sin\theta)\,\dot{\theta}$$

$\|(\dot{x}, \dot{y})\|^2 = \dot{x}^2 + \dot{y}^2$

polarisation identity: $\|x+y\|^2 - \|x\|^2 - \|y\|^2 = 2\langle x, y\rangle$
($\|x+y\|^2 - \|x-y\|^2 = 4\langle x, y\rangle$)

on $\mathbb{R}^2$ standard metric maps vector $\begin{pmatrix} a \\ b \end{pmatrix}$ to functional $a\,dx + b\,dy$

Here $dx\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \Delta x$, $dy\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \Delta y$

$\langle g\binom{a}{b}, \binom{c}{d}\rangle = \langle a\,dx + b\,dy, \binom{c}{d}\rangle = ac + bd$

$\|(\dot{x}, \dot{y})\|^2 = \dot{x}^2 + \dot{y}^2 = R^2\,\dot{\theta}^2$

$x = R\theta^2, \quad y = \theta^3$

$\dot{x}^2 + \dot{y}^2 = \left((2R\theta)^2 + (3\theta^2)^2\right)\dot{\theta}^2$

$x = r\cos\theta, \quad y = r\sin\theta$

$\dot{x} = \dot{r}\cos\theta - r\sin\theta\,\dot{\theta} \; ; \; \dot{y} = \dot{r}\sin\theta + r\cos\theta\,\dot{\theta}$

so $\dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\theta}^2$

metric does not map $\binom{\Delta r}{\Delta\theta} \to \Delta r\, dr +$

$g\binom{a}{b} = a\,dx + b\,dy$

$dx = dr\cos\theta - r\sin\theta\,d\theta$

$dy = dr\sin\theta + r\cos\theta\,d\theta$