Math 928, lecture 7, 28/1/2025

Last time: Calculus of variations

Methods for solving optimization problems with objective function

$$S(\gamma) = \int_a^b L(\gamma(t), \dot\gamma(t); t)\, dt$$

over functions $\gamma : [a,b] \to \mathbb{E}^{dN}$, $\gamma \in C^1$

$L : \mathbb{E}^{dN} \times \mathbb{R}^{dN} \times \mathbb{R} \to \mathbb{R}$, $L \in C^1$

Goal: Find critical pts.

Direct method: show S has min in some function space.

Indirect method: Find sufficient conditions for critical pts, solve for them

For our purposes don't need to differentiate S wrt γ, instead take directional derivatives in enough directions ("variational derivative")

Formal calculation: Suppose $S(\gamma)$ minimal for all γ st. $\gamma(t_0) = a$, $\gamma(t_1) = b$.

Idea: Consider $S(\gamma+\varepsilon\eta)-S(\gamma)$, $\eta\in C_c^\infty(a,b)$
$\varepsilon\to 0$

Then $L(\gamma(t)+\varepsilon\eta(t),\dot\gamma(t)+\varepsilon\dot\eta(t);t)-L(\gamma(t),\dot\gamma(t);t)$
$$=\varepsilon\left\langle\frac{\partial L}{\partial q},\eta(t)\right\rangle+\varepsilon\left\langle\frac{\partial L}{\partial\dot q},\dot\eta(t)\right\rangle+O(\varepsilon^2)$$

evaluated at $(\gamma(t),\dot\gamma(t);t)$.

so
$$S(\gamma+\varepsilon\eta)-S(\gamma)=\varepsilon\int_{t_0}^{t_1}\left[\left\langle\frac{\partial L}{\partial q},\eta(t)\right\rangle+\left\langle\frac{\partial L}{\partial\dot q},\dot\eta(t)\right\rangle\right]dt+O(\varepsilon^2)$$
$$\underset{\text{by parts}}{=}\varepsilon\int_{t_0}^{t_1}\underbrace{\left\langle\frac{\partial L}{\partial q}-\frac{d}{dt}\frac{\partial L}{\partial\dot q},\eta(t)\right\rangle dt}+O(\varepsilon^2)$$

$\eta(t_0)=\eta(t_1)=0$

If $S(\gamma)$ minimal, must have integral vanishes

Arbitrary choice of η $\Rightarrow$

$$\frac{\partial L}{\partial q}(\gamma(t),\dot\gamma(t);t)-\frac{d}{dt}\left(\frac{\partial L}{\partial\dot q}(\gamma(t),\dot\gamma(t);t)\right)=0$$

"Euler-Lagrange Equation"

2^{nd} order ODE. Typically has solutions

Remark: Solve ODE from initial conditions
here we have boundary values.

Formalization:

Lemma: (1) Let $f \in C(a,b)$, suppose for all $\eta \in C_c^\infty(a,b)$, $\eta \geq 0$ we have $\int_a^b f\eta \, dt \geq 0$. Then $f \geq 0$.

(2) Let $f \in L^1(a,b)$, same assumption. Then $f \geq 0$ a.e.

Pf: (1) Say $f(s) < 0$. Then have interval $J \subset\subset (a,b)$ on which $f \leq -\delta < 0$. Let η be positive supported in J. Then $\int_a^b f\eta \, dt = \int_J f\eta \leq -\delta \int_J \eta \, dt < 0$.

(2) Let $d\mu = f\,dt$ be the measure on $[a,b]$ with density f wrt Lebesgue. $\mu(\eta) = \int_a^b f\eta \, dt$. (Riesz Representation thm): since $\mu(\eta) \geq 0$ if $\eta \geq 0$, μ is a positive measure $\Rightarrow \frac{d\mu}{dt} = f \geq 0$. ∎

Taste of "direct" arguments.

Def: Say L is "standard" if

$$L(q,\dot{q};t) = \frac{1}{2}\langle M(q,t)\dot{q}, \dot{q}\rangle - U(q,t)$$

where $M = M(q,t)$ is symmetric, $M \geq \mu > 0$ for some constant, U "nice".

Fact: If M = mass, U = potential, then Euler-Lagrange = "Newton"

Def: For nice enough $\gamma: I \to \mathbb{R}^n$, define

$$\|\gamma\|^2_{H^k} = \sum_{i=0}^{k} \|\gamma^{(i)}\|^2_{L^2(I)}$$ ← Sobolev norm

$$\|\gamma\|^2_{H^0} = \|\gamma\|^2_{L^2} = \int_I |\gamma(t)|^2 dt.$$

$H^k(I;\mathbb{R}^n)$ = completion of $C^\infty_c(I;\mathbb{R}^n)$ wrt norm

$\iff \{ f \in L^2 \mid k^{th}$ distributional derivative also in $L^2 \}$

Ex: Equivalent norm is $\sqrt{\|\gamma\|^2_{L^2} + \|\gamma^{(k)}\|^2_{L^2}}$.

On $\mathbb{R}$ equivalent to $\|(1+|\xi|)^k \hat{f}\|_2$. ξ

Ex: $H^1 \supseteq \{\gamma : \gamma \text{ is piecewise } C^1\}$

Ex: $H^k \subset C^{k-1}(I)$, inclusion is compact.

Lemma: (Poincaré inequality) Let $u: I \to \mathbb{R}^n$ be diff with $u(t_0) = u(t_1) = 0$. Then

$$\int_{t_0}^{t_1} |u|^2 dt \le 2\left(\frac{t_1 - t_0}{2}\right)^2 \int_{t_0}^{t_1} |\dot{u}|^2 dt$$

Pf: Wlog $I = [-\Delta, \Delta]$. Then

$$\int_{-\Delta}^{\Delta} |u|^2 dt = \left[t|u|^2\right]_{-\Delta}^{\Delta} - 2\int_{-\Delta}^{\Delta} t\, u\, \dot{u}\, dt$$

$$\leq 2\Delta \cdot \left(\int_{-\Delta}^{\Delta} |u|^2 dt\right)^{1/2} \left(\int_{-\Delta}^{\Delta} |\dot u|^2 dt\right)^{1/2}. \blacksquare$$

<u>Lemma:</u> (Coercivity) Suppose $U(q) \leq A + C|q|^2$

$\forall \varepsilon > 0 \ \exists \delta > 0$ s.t.

$$\int_{t_0}^{t_0+\delta} U(q,t)\,dt \leq B + \varepsilon \int_{t_0}^{t_0+\delta} |\dot q|^2 dt$$

<u>Pf:</u> let $\bar q$ linearly interpolate $q(t_0) = a$, $q(t_1) = b$

Let $u = q - \bar q$. Then

$$\dot u = \dot q - \dot{\bar q} = \dot q - \frac{b-a}{\delta}$$

$$\text{so } |\dot u|^2 \leq 2|\dot q|^2 + 2\frac{(b-a)^2}{\delta^2}$$

So $\quad U(q) \leq A + C|q|^2 \leq A + 2C|\bar q|^2 + 2C|u|^2$

So $\quad \displaystyle\int_{t_0}^{t_0+\delta} U(q)\,dt \leq A\delta + 2C\delta(|a|^2 + |b|^2) + C\delta^2 \int_{t_0}^{t_0+\delta} |\dot u|^2 dt$

$$\leq A\delta + 2C\delta(|a|^2 + |b|^2) + 2C\delta\,|a-b|^2$$

$$+ 2C\delta^2 \int_{t_0}^{t_0+\delta} |\dot q|^2 dt.$$

Take δ small so that $2C\delta^2 < \varepsilon$. $\blacksquare$

<u>Corollary:</u> If $\varepsilon < \mu$, (1) $S(\gamma)$ bounded below

(2) Sublevel sets bounded in H^1.

(if $S(\gamma) \leq T$, $\|\gamma\|_{H^1} \leq f(T)$)

Hamilton's Principle: Solutions to Newton's equations are critical pts of the action

(if δ small enough set "least action")

$$\frac{1}{2}\langle M\underline{v}(t+dt), \underline{v}(t+dt)\rangle \approx \frac{1}{2}\langle M\underline{v}, \underline{v}\rangle + \frac{1}{2}\langle M\underline{v}, \underline{a}\cdot dt\rangle + \frac{1}{2}\langle M\underline{a}\,dt, \underline{v}\rangle + \cancel{\frac{1}{2}\langle M\underline{a}, \underline{a}\rangle (dt)^2}$$

to 1^{st} order

$\underline{v}(t+dt) \approx \underline{v}(t) + \underline{a}(t)dt$

$$\approx \frac{1}{2}\langle M\underline{v}, \underline{v}\rangle + \left(\frac{1}{2}\langle M\underline{a}, \underline{v}\rangle + \frac{1}{2}\langle M\underline{a}, \underline{v}\rangle\right)dt$$

$$\approx \frac{1}{2}\langle M\underline{v}, \underline{v}\rangle + \langle M\underline{a}, \underline{v}\rangle\, dt$$

$$\frac{d(T)}{dt} = \langle M\underline{a}, \underline{v}\rangle = \langle \vec{F}, \underline{v}\rangle$$

(if $\ddot{y} = f$ then $(\dot{y})^{2\,\cdot} = 2\dot{y}\,\ddot{y} = 2\dot{y}f$)

say $f = f(y)$ then

$$\dot{y}\ddot{y} = f(y)\dot{y}$$

$\frac{d}{dt}\left(\frac{1}{2}\dot{y}^2\right)$ $\quad \frac{d}{dt}U(y)$ $\quad$ if $\frac{dU}{dy} = f$

Leibnitz rule $\quad (fg)' = f'g + fg'$

$$\langle f(x+\Delta x), g(x+\Delta x)\rangle \approx \ldots$$

$f(\underline{v}) = \underline{v}^T A\,\underline{v}$ $\quad$ A matrix

$$f(\underline{v}+\underline{\delta}) = (\underline{v}+\underline{\delta})^T A(\underline{v}+\underline{\delta}) = \underline{v}^T A\underline{v} + \underline{\delta}^T A\underline{v} + \underline{v}^T A\underline{\delta} + \underline{\delta}^T A\underline{\delta}$$

$$= \underline{v}^T A \underline{v} + \underline{v}^T A^T \underline{\delta} + \underline{v}^T A \underline{\delta} + O(|\delta|^2)$$

to 1st order

$$= f(\underline{v}) + \left(\underline{v}^T (A + A^T)\right) \cdot \underline{\delta} + O(|\delta|^2)$$