Math 428, lecture 8, 30/1/2024

Last time: Direct approach: introduced Sobolev space

$$\|\gamma\|^2_{H^1} = \int |\gamma(t)|^2 dt + \int |\dot\gamma(t)|^2 dt$$

Standard action $S(\gamma) = \int_{t_0}^{t_1} \left(\frac{1}{2}\langle M\dot\gamma, \dot\gamma\rangle - U(\gamma)\right) dt$ can be interpreted in that space.

Proved: if t_1 close enough to t_0, if U is sub-quadratic then minimizing sequences bounded in H^1.

Observation: Formulation is intrinsic, does not depend on any co-ordinates on $\mathbb{E}^n$ where γ is valued

Better: With constraints, the physical path is still a critical point, after restricting to space of constrained paths

Eg. (Holonomic constraints): physical path is critical for $S(\gamma)$, $\gamma \in \{\text{paths in } X\}$ fixed endpoints

Aside: what do we mean by "deformation" if X not vsp?

Solution 1: work in co-ords $q: V \to \mathbb{R}^n$
$V \subseteq X$
then think of co-ord path

$$q(t) = q \circ \gamma,$$

which is valued in vsp

Solution 2: If γ' close to γ, $\gamma'(t) - \gamma(t) \approx$ vector in $T_{\gamma(t)} X$.

⇝ "Banach manifolds"

Hamilton's principle: For any t_0, t_1 the physical path
$\gamma : [t_0, t_1] \to X$
is critical for $S(\gamma) = \int_{t_0}^{t_1} L \, dt$
among paths into X with same endpoints.

L = Lagrangian = $T - U$

kinetic energy

potential energy

recall: $\Rightarrow \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}$

Example: In $\mathbb{E}^d$, let $T = \frac{1}{2} \sum_j m_j \dot{x}_j^2$
then the equation reads

$$M \ddot{x} = - \frac{\partial U}{\partial x}$$

Examples in 2d gravity in y direction

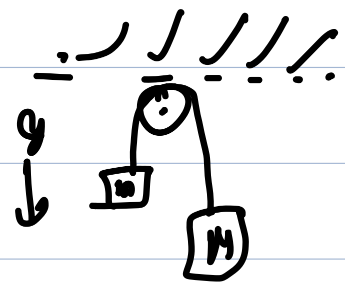
$$M\ddot{x} = m(0,-g)$$

rotate by $\frac{\pi}{4}$. $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{X+Y}{\sqrt{2}} \\ \frac{X-Y}{\sqrt{2}} \end{pmatrix}$ $\quad X = \frac{x+y}{\sqrt{2}}$

$\quad Y = \frac{x-y}{\sqrt{2}}$

$$m(\ddot{X}, \ddot{Y}) = m\left(-\frac{g}{\sqrt{2}}, \frac{g}{\sqrt{2}}\right)$$

$$m(\ddot{x}, \ddot{y}) = m(0, -g)$$

Example:  $\subseteq \mathbb{E}^2$

first mass at y second at Y

constraints $y + Y = \text{const}$

$$\dot{y} + \dot{Y} = 0 \Rightarrow \dot{Y} = -\dot{y}$$

$\Rightarrow$

$$T = \frac{1}{2} m \dot{y}^2 + \frac{1}{2} M \dot{y}^2$$

$$U = -mgy - MgY = (M-m)gy + \text{const}$$

so $\quad L = \frac{1}{2}(m+M)\dot{y}^2 - (M-m)gy$

$\Rightarrow$

$$\frac{d}{dt}(m+M)\dot{y} = -(M-m)g$$

$$\Rightarrow \ddot{y} = -\frac{M-m}{m+M} g$$

Examples Central potential

One particle in $\mathbb{R}^d$, potential $U(r)$, r = distance to 0.

$$L = \frac{1}{2} m |\dot{x}|^2 - U(|x|)$$
$$= \frac{1}{2} m \sum_{i=1}^{d} \dot{x}_i^2 - U\left(\left(\sum_{i=1}^{d} x_i^2\right)^{1/2}\right)$$

instead, write $x = r \cdot \hat{\theta}$, $r \in (0, \infty)$, $\hat{\theta} \in S^{n-1}$

Ex: $L = \frac{1}{2} m \left(\dot{r}^2 + r^2 |\dot{\theta}|^2\right) - U(r)$

metric of round sphere

$$\Rightarrow m \frac{d}{dt}\left(\dot{r}, r^2 \dot{\theta}\right) = \underbrace{-\left(\frac{\partial U}{\partial r}, 0\right) + \left(m r |\dot{\theta}|^2, 0\right)}_{\frac{\partial L}{\partial q}}$$

$\Downarrow$

$$\begin{cases} m \ddot{r} = -\frac{\partial U}{\partial r} + m r |\dot{\theta}|^2 \\ m r^2 \ddot{\theta} + 2 m r \dot{r} \dot{\theta} = 0 \quad \left(\frac{d}{dt}(r^2 \dot{\theta}) = 0\right) \end{cases}$$

Example: $d = 2$ then

$$L = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2\right) - U(r)$$

$$\Rightarrow \quad \ddot{r} = -\frac{1}{m}\frac{\partial U}{\partial r} + r \dot{\theta}^2 ; \quad r^2 \dot{\theta} = \text{Const.}$$

Observe (Kepler): $r^2 \Delta\theta$ = area of wedge

if $r^2 \dot\theta = M$ then $r\dot\theta^2 = \frac{M^2}{r^3}$

so $\ddot r = -\frac{1}{m}\frac{\partial U}{\partial r} + \frac{M^2}{r^3}$ ODE for r.

mult by $\dot r$, integrate dt

$$\frac{1}{2}\dot r^2 = -\frac{1}{m}U - \frac{2M^2}{r^2}$$

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point is $\theta = (x, y, \sqrt{1-x^2-y^2})$

$$\dot\theta = \left(\dot x, \dot y, -\frac{x\dot x + y\dot y}{\sqrt{}}\right)$$

$$|\dot\theta|^2 = \dot x^2 + \dot y^2 + \frac{(x\dot x + y\dot y)^2}{1-x^2-y^2}$$

$(r\sin\varphi\cos\theta, r\sin\varphi\sin\theta, r\cos\varphi)$

$$\dot\theta = (\cos\varphi\cos\theta \cdot \dot\varphi - \sin\varphi\sin\theta\,\dot\theta, \dots)$$