Math 428, Lecture 3

Recently: to our physical system we associate
- Lagrangian $L: TX \times \mathbb{R} \to \mathbb{R}$

(default: $L = \underset{\text{kinetic term}}{T} - \underset{\text{potential}}{U}$)

$\Rightarrow$ Action $S(\gamma) = \int_{t_0}^{t_1} L(\gamma(t), \dot\gamma(t); t)\, dt$

Hamilton's principle: S is stationary about physical path $\gamma: [t_0, t_1] \to X$

(Direct method: if t_1 close enough to t_0, U does not grow too fast, then S actually has minimizers in H^1)

Recall: TX ("tangent bundle") is the space of states of motion:

$$TX = \left\{ (x,v) : \begin{array}{l} x \in X \\ v \in T_x X \end{array} \right\} = \coprod_{x \in X} T_x X.$$

Hamilton's principle ⇒ "variational derivative":
to first order

$$S(\gamma+\epsilon\eta) \approx S(0) + \varepsilon \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot q}\right)\eta = 0$$

$$\delta L = \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot q} \quad \text{along } \gamma(t)$$

⇒ Euler-Lagrange Equations $\frac{\partial L}{\partial q} = \frac{d}{dt}\frac{\partial L}{\partial \dot q}$

in co-ords $q = (q_\alpha)_{\alpha=1}^{n}$ set

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q_\alpha} = \frac{\partial L}{\partial q_\alpha}$$

Say $L = T - U = \frac{1}{2}m\dot x^2 - U(x)$

see: $\frac{\partial L}{\partial \dot x} = m\dot x$, $\frac{\partial L}{\partial x} = -\frac{dU}{dx}$

⇒ Call $\frac{\partial L}{\partial \dot q_\alpha}$ the generalized momentum associated to co-ord α.

Call $\frac{\partial L}{\partial q_\alpha}$ the generalized force.

$p_\alpha = \frac{\partial L}{\partial \dot q_\alpha}$

(Example: q = co-ordinates on sphere S^{d-1}, generalized momentum = "angular momentum"
" force = " torque")

Observation: if $\frac{\partial L}{\partial q_\alpha} = 0$ (say q_α is cyclic) then

$$\frac{d}{dt}(p_\alpha) = 0,$$ i.e. p_α is conserved = constant of the motion.

Example: $H = \dot{q}\frac{\partial L}{\partial \dot{q}} - L = \sum_\alpha \dot{q}_\alpha \frac{\partial L}{\partial \dot{q}_\alpha} - L$

$$\frac{dH}{dt} = \ddot{q}\frac{\partial L}{\partial \dot{q}} + \dot{q}\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{dL}{dt}$$

$$= \cancel{\ddot{q}\frac{\partial L}{\partial \dot{q}}} + \cancel{\dot{q}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q}\dot{q}} - \cancel{\frac{\partial L}{\partial \dot{q}}\ddot{q}} - \frac{\partial L}{\partial t}$$

E-L

$$\Rightarrow \frac{dH}{dt} = -\frac{\partial L}{\partial t},$$ hence is constant if L is indep of t.

Call H the energy of the system

Ex: if $L = T - V$, $T = \frac{1}{2}\dot{q}M(q,t)\dot{q}$
$U = U(q)$
then $L = T + U$.

Debt: We formalized this for $X = \mathbb{E}^{dN}$.
What about constraints?

Answer: Method of Lagrange multipliers applies here as well

Say we have constraints $f_i(q;t) = 0$
Then constrained minima / critical pts optimize

$$\int_{t_0}^{t_1} \Big(L + \sum_i \lambda_i(t) f_i\Big) dt \quad (\text{over } \gamma, \{\lambda_i\})$$

So $\quad \delta \Big(L + \sum_i \lambda_i f_i\Big)$

$$= \frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot q} + \sum_i \lambda_i \frac{\partial f_i}{\partial q}$$

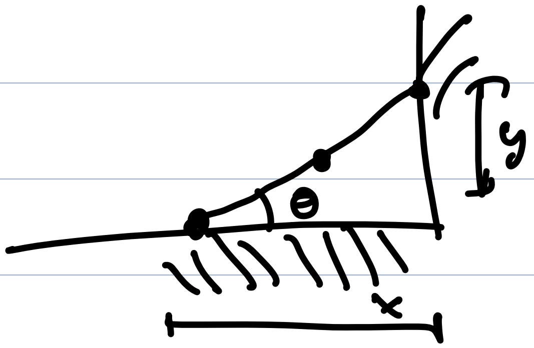
$$\Rightarrow \begin{cases} \dfrac{d}{dt} p = \dfrac{\partial L}{\partial q} + \displaystyle\sum_i \lambda_i \frac{\partial f_i}{\partial q} \leftarrow \text{constraint force} \\ f_i = 0 \leftarrow \text{constraints} \end{cases}$$

check: Along X, just have usual E-L equations (by def'n $T_x X = \ker \frac{\partial f}{\partial q}$)

Remark: Can also apply this to constraints of the form $f(x, \dot{x}; t) = 0$
(e.g. rolling without slipping)

Then the constraint force is $\sum_i \left[\lambda_i \left(\frac{\partial f_i}{\partial q} - \frac{d}{dt} \frac{\partial f_i}{\partial \dot{q}} \right) + \dot{\lambda}_i \frac{\partial f_i}{\partial \dot{q}} \right].$

HW:



ladder resting on wall, wall floor smooth.

mass m, length L. angle θ, x, y

Kinetic energy: $\frac{1}{2} m \left(\frac{1}{4} \dot{x}^2 + \frac{1}{4} \dot{y}^2 \right) + \frac{1}{2} I \cdot \dot{\theta}^2$

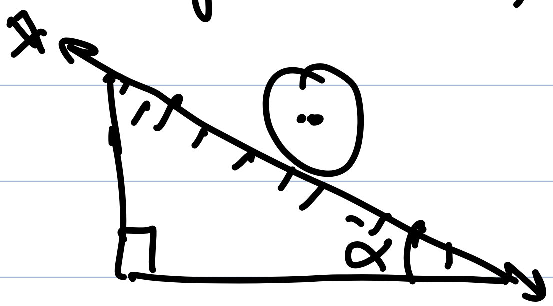
Potential energy: $\frac{1}{2} m g y$

constraints: $y = L \sin\theta$, $x = L \cos\theta$.

come off wall when assoc constraint force is zero.

$\vdots$

Example: (Rolling without slipping)



hoop
rolling without slipping
on fixed incline

θ angle of rotation, $r\,d\theta = dx \Rightarrow x = r\theta$

$$L = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 - m g x \sin\alpha$$

$$= m \dot{x}^2 - m g x \sin\alpha$$

on X

With constraint: $\delta\left(\frac{1}{2} m \dot{x}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 - m g x \sin\alpha + \lambda(x - r\theta)\right)$

$$\begin{cases} \frac{d}{dt}(m\dot{x}) = -mg\sin\alpha + \lambda \\ \frac{d}{dt}(m r^2 \dot{\theta}) = -\lambda r \\ x = r\theta . \end{cases}$$

set: $\ddot{x} = r\ddot{\theta}$; $\rightarrow m r^2 \ddot{\theta} = -\lambda r$

$m\ddot{x} = -\lambda \leftarrow$ at each t, $m\ddot{x}(t) = -\lambda(t)$

$$2m\ddot{x} = -mg\sin\alpha$$

$$\Rightarrow \boxed{\ddot{x} = -\tfrac{1}{2} g\sin\alpha} \quad \Rightarrow \boxed{\ddot{\theta} = -\tfrac{1}{2}\frac{g\sin\alpha}{r}}$$

$$\lambda = -m\ddot{x} = \tfrac{1}{2} mg\sin\alpha$$

friction force between hoop & incline

Remark: with constraint $\frac{x}{r} - \theta = 0$ the force would be $\frac{1}{2} mgr\sin\alpha$, i.e. the torque by friction on hoop

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Conserved quantities: Noether's Theorem

Observation: symmetry $\Leftrightarrow$ conservation laws

Def: a one-parameter group acting on X is a smooth map $g: \mathbb{R} \times X \to X$

write $g_r(x)$ rather that $g(r;x)$.

st: $g_0 = \mathrm{id}_X$, $\quad g_{r_1+r_2} = g_{r_1} \circ g_{r_2}$

Example: (1) $\mathfrak{X} = \mathbb{E}^d$, $g_r(x) = x + rv$

(2) $\mathfrak{X} = \mathbb{R}^2$, $g_\theta \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

we then have $d(g_r)|_x : T_x X \to T_{g_r(x)} X$.

Also have infinitesimal transformation

$$\frac{dg}{dr}\Big|_{\substack{x \\ r=0}} \in T_x \mathfrak{X} \leftarrow$$ start at x, move to $g_\varepsilon(x)$, ε small, in which direction are we going?

Theorem: (Noether 1918) Along the motion, suppose $L \circ g_r = L$.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \cdot \frac{dg}{dr}\Big|_{\substack{r(t) \\ r=0}}\right) = 0$$

(translation invariance $\Rightarrow$ conservation of linear momentum)

(rotation invariance $\Rightarrow$ same for angular momentum)