Math 428, lecture 10

Last time: Momentum $p = \frac{\partial \mathcal{L}}{\partial v}$, $p_j = \frac{\partial \mathcal{L}}{\partial \dot q_j}$

$L = L(x, v, t)$ $\quad$ $L = L(q_1, .., q_n, \dot q_1, .., \dot q_n, t)$

$(x, v) \in TX$ $\quad$ co-ords

Force $\frac{\partial L}{\partial x}$; $F_j = \frac{\partial L}{\partial q_j}$.

$\left(\text{E-L}: \frac{d}{dt} p_j = F_j \right)$

Observed if q_j is cyclic: $\frac{\partial \mathcal{L}}{\partial q_j} = 0$, p_j is conserved.

Also Beltrami identity $\frac{d}{dt} \mathcal{E} = -\frac{\partial \mathcal{L}}{\partial t}$

with

$$\mathcal{E} = \frac{\partial \mathcal{L}}{\partial v} v - L = \sum_j \frac{\partial \mathcal{L}}{\partial \dot q_j} \dot q_j - L$$

Today: (1) 2 examples

(2) Noether's Theorem

(3) Angular momentum

§1. Example:

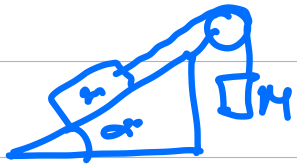
(1) particle in 2d, downward gravity:

$$L = \frac{1}{2} m (\dot x^2 + \dot y^2) + mgy$$

x is cyclic

But set $z = x + y$, now $L = \frac{1}{2} m (\dot x^2 + (\dot z - \dot x)^2) + mg(z - x)$

now x not cyclic $\Rightarrow$ cyclicity is property of coord system not just of x.

(2) 

relative to the wedge,
M is at (a, y)
m is at $(c, d) - y(\cos\alpha, \sin\alpha)$

Say we accelerate wedge: at $X(t)$ at time t.
Then M is at $(a + X(t), y)$
m is at $(c + X(t), d) - y(\cos\alpha, \sin\alpha)$

$$\Rightarrow L = \frac{1}{2}M(\dot{X}^2 + \dot{y}^2) + \frac{1}{2}m\left((\dot{X} - \dot{y}\cos\alpha)^2 + \dot{y}^2\sin\alpha\right) + Mgy - mg\sin\alpha\, y.$$

$$\frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}M\dot{y}^2 + \frac{1}{2}(M+m)\dot{y}^2 - m\cos\alpha\,\dot{x}\dot{y} + (M - m\sin\alpha)gy$$

$\Rightarrow$ Equations of motion, ..

Observation: $\frac{1}{2}M\dot{x}^2$ = function of time

So instead $\tilde{L} = L - \frac{1}{2}(M+m)\dot{x}^2$ has same EoM

Def: given $f: X \times \mathbb{E}' \to \mathbb{R}$, the total derivative of f is

$\frac{df}{dt}$ $\quad (x, v, t) \mapsto \langle \frac{\partial f}{\partial x}, v \rangle + \frac{\partial f}{\partial t}$

Lemma: $\frac{d}{dt} f(\gamma(t); t) =$ total derivative $(\gamma(t), \dot\gamma(t); t)$

$$\Rightarrow \int_{t_0}^{t_1} \frac{df}{dt}(\gamma, \dot\gamma, t)\, dt = f(\gamma(t_1), t_1) - f(\gamma(t_2); t_2)$$

Cor: If $\hat{L} = L + \frac{df}{dt}$, then $\hat{S}(\gamma) = S(\gamma) + f(b, t_1) - f(a, t_2)$

for paths connecting $(t_0, a) \rightsquigarrow (t_1, b)$

Example: $q\dot{q} = \frac{d}{dt} q^2.$

Symmetry

Def: A one-parameter group is a smooth family of smooth maps $g_r : X \times \mathbb{E}^1 \to X \times \mathbb{E}^1$, s.t.: $g_0(x,t) = (x,t)$

$$g_{r+s} = g_r \circ g_s$$

Def: g_r is a symmetry of L, equiv. L is invariant by g_r, if for all r, $L \circ g_r - L$ is a total derivative.

$$(L \circ g_r)(x, v, t) = L\left(g_r(x,t), \left.\frac{\partial g_r}{\partial x}\right|_x (v)\right)$$

(equiv, action is $\hat{S}(\gamma) = \int L(g_r \circ \gamma)\, dt$)

For fixed x, t, $r \mapsto g_r(x;t)$ is a map $\mathbb{R} \to X \times E^1$
Can diff wrt r, get derivative $(g_r'(x,t), \tau_r'(x,t))$
write $g'(x,t)$, $\tau'(x,t)$ for values at $r=0$

<u>Theorem:</u> Suppose $\{g_r\}$ is a symmetry of L. Then
(Noether 1918)

$$\langle \frac{\partial L}{\partial v}, g'(x,t) \rangle - \tau'(x,t) E$$

is conserved

<u>Examples</u> $g_r(x,t) = (x, t+r)$
$g'(x,t) = 0$, $\tau'(x,t) = 1$ $\Rightarrow$ conservation of energy.

Consider first, case $g_r : X \to X$ (no change in time)
write $g'(x) = \frac{d}{dr}\Big|_{r=0} (g_r(x)) \in T_x X$.

<u>lemma:</u> the curves $r \mapsto g_r(x)$ are the integral curves of this vector field, i.e. the solutions to ODE $\frac{dy}{dr} = g'(y(r))$

<u>Pf:</u> $g_{r+\varepsilon}(x) = g_\varepsilon(g_r(x)) \Rightarrow \frac{dg_r}{dr}(x) = g'(g_r(x))$.
By uniqueness, $g_r(x)$ is the solution to ODE.

Theorem: (Weak Noether's Thm) If $g_r(x)$ is a symmetry; $\langle \frac{\partial L}{\partial v}, g' \rangle$ is conserved

$L \circ g_r = L$

Pf: $0 = \frac{d}{dr} S(g_r \circ \gamma) = \frac{d}{dr} \int_{t_0}^{t_1} L(g_r(\gamma(t)), dg_r(\dot{\gamma}), t)\, dt$

$$= \int_{t_0}^{t_1} \left[\langle \frac{\partial L}{\partial x}, g'(g_r(\gamma(t))) \rangle + \langle \frac{\partial L}{\partial v}, \frac{d}{dr}(d_x g_r(\dot{\gamma}(t))) \rangle \right] dt$$

At $r=0$, $0 = \frac{d}{dr}\Big|_{r=0} S(g_r \circ \gamma) = \int_{t_0}^{t_1} \left[\langle \frac{\partial L}{\partial x}, g'(x) \rangle + \langle \frac{\partial L}{\partial v}, \frac{d}{dr} d_x g_r(\dot{\gamma}(t)) \rangle \right] dt$

By chain rule, $\frac{d}{dr}(d_x g_r(\dot{\gamma}(t)) = \frac{d}{dt}(g'(g_r(\gamma(t)))$

set $r=0$, integrate by parts:

$$0 = \int_{t_0}^{t_1} \left[\langle \frac{\partial L}{\partial x}, g'(x) \rangle - \langle \frac{d}{dt} \frac{\partial L}{\partial v}, g'(x) \rangle \right] dt + \left[\langle \frac{\partial L}{\partial v}, g'(v) \rangle \right]_{t_0}^{t_1}$$

$$\Rightarrow \left[\frac{\partial L}{\partial v} \cdot g'(v) \right]_{(a, t_0)}^{(b, t_1)} = - \int_{t_0}^{t_1} \langle \underbrace{\left(\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial v} \right)}_{= 0 \text{ EoM}}, g' \rangle dt$$

∎

Example: if q_j is cyclic, $g_r(q_1, \dots, q_n) = (q_1, \dots, q_n) + r \underline{e}_j$. is a symmetry.

§3. Rotations

in 2d $\quad g_\theta\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$

① linear algebra

Equip $\mathbb{R}^d$ with inner pdt, Euclidean distance. Let $O(d)$ be the group of symmetries = rigid motions fixing origin.

Chaos
Gleick

lemma: each $g \in O(d)$ is a linear map, satisfies $g^*g = Id$ $\quad$ ($g \in M_d(\mathbb{R})$, g^* = transpose)

and $O(d) = \{g \in M_d(\mathbb{R}) : g^*g = I_d\}$.

Observe: g^*g symmetric, only upper triangle matters
set $\dfrac{d(d+1)}{2}$ constraints

lemma: $F(g) = g^*g$ is non-degenerate on $O(d)$
Tangent space at I_d is $so(d) = \{X \in M_n(\mathbb{R}) \mid X^* + X = 0\}$

Pf: Given deformation $Y \in M_d(\mathbb{R})$, have:

$$F(g+Y) = (g^* + Y^*)(g+Y)$$
$$= g^*g + (g^*Y + Y^*g) + O(Y^2)$$
$$\Rightarrow dF_g(Y) = g^*Y + Y^*g$$

when g is invertible, $\{g^*Y\}_Y = M_d(\mathbb{R})$

so $\{g^*Y + Y^*g\}$ = symmetric matrices

has dim $\frac{d(d+1)}{2}$. At $g = id$ $dF_*(Y) = Y^* + Y$

$\ker = \{Y \mid Y^* = -Y\}$.

<u>Def:</u> Call $X \in so(d)$ infinitesimal rotations.

$\dim_{\mathbb{R}} so(d) = \dim_{\mathbb{R}} O(d) = \frac{d(d-1)}{2}$