Math 428, lecture 11, 11/2/25

Last time: Noether's theorem

$g_r : X \times E'$'s 1-parameter group $\quad g_0 = id_{X\times E'}$ $(g_r \circ g_s = g_{r+s})$

Theorem: If $(g_r)_r$ is a symmetry of L then

$$\left\langle \frac{\partial L}{\partial v}, g' \right\rangle - \tau' \left(\frac{\partial L}{\partial v} v - L \right)$$

is conserved $\quad \left.\frac{dg}{dr}\right|_{\substack{r=0\\(x,t)}} = (g', \tau') \in T_x X \times \underset{T_t E'}{\mathbb{R}}$.

Rotations

Equip $\mathbb{R}^d$ with Euclidean inner prod, metric. $O(d) =$ rigid motions fixing $\underline{0}$

so $O(d) = \{ g \in M_d(\mathbb{R}) \mid g^* g = I_d \}$ $\quad g^* =$ transpose

Then $O(d) =$ level set of $F(g) = g^* g$ (only upper triangle)

Lemma: F is non-degenerate on $O(d)$. $\quad \boxed{SO(d) = \{ g \in O(d) : \det g = 1 \}}$

$$so(d) = T_1 O(d) = \{ X \in M_d(\mathbb{R}) : X^* + X = 0 \}$$

Def: Call $X \in so(d)$ an infinitesimal rotation

Today: rotations & angular momentum.

Def: let $X \in M_d(\mathbb{K})$, the matrix exponential is

$$\exp(X) \overset{\text{def}}{=} \sum_{k=0}^{\infty} \frac{1}{k!} X^k$$

the matrix logarithm is $\log g = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k} (g - I_d)^k$

lemma: Entries of X^k grow at most exponentially in k

so $\exp(X)$ converges abs. for all X, $\log g$ converges absolutely if g close enough to 1, $\log(\exp(X)) = X$, $\exp(\log(g)) = g$

is X, g small enough, and if X, Y commute $\exp(X)\exp(Y) = \exp(X+Y)$.

Proof: same power series arguments as in Calc 1. ∎

Cor: exp, log identify nbd of 0 in $M_n(\mathbb{K}) = \mathfrak{gl}_n \mathbb{K}$ with nbd of 1 in $GL_n(\mathbb{K})$

lemma: For small enough nbd $V \subset O(d)$, $\log : V \to \mathfrak{so}(d)$ is a coordinate system.

corresponding parametrization is $\exp|_{\mathfrak{so}(d)}$

Pf: If $g \in SO(d)$, $g^* = g^{-1}$, commutes with g.

$\Rightarrow \log(g^* g) = \log(g^*) + \log g$ (if g small enough)

take U small enough s.t. $\log$ converges on U,
(take V small enough s.t. $V = V^{-1} \subset U$, $VV \subset U$)

so $(\log g)^* + \log g = \log(g^*) + \log g = \log(g^* g) = 0$.

conversely if $X \in so(d)$ $X^* = -X$ commutes with X,
then

$\exp(X)^* \exp(X) = \exp(X^*) \exp(X) = \exp(X^* + X) = 0$

Also $\log(I + X + O(X^2)) = X + O(X^2)$
$\Rightarrow \log$ has full rank, $d_I(\log) = id$

Cor: For $X \in so(d)$, $\{\exp(rX)\}_{r \in \mathbb{R}} \subset O(d)$ is a 1-parameter group acting on $\mathbb{R}^d$

Fact: Write $\mathbb{R}^d \cong (\mathbb{R}^2)^{d/2}$, or $(\mathbb{R}^2)^{\frac{d-1}{2}} \oplus \mathbb{R}$
let $X_i = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ in ith summand, then X_i commute,

$\{(\exp(rX_i))_r\}_{i=1}^{\lfloor d/2 \rfloor}$ is a maximal commuting family of 1-param subgroups, $\{\exp(\sum_{i=1}^{\lfloor d/2 \rfloor} r_i X_i)\}$ is a max'l torus

Angular momentum

Say in $\mathbb{E}^d$ we have $L = \frac{1}{2}\sum_{j=1}^{N} m_j v_j^2 + U(x)$

Each $g \in O(d)$ acts on $\mathbb{E}^d$, we have $\frac{d}{dt}(g x_j) = g v_j$
so $|g v_j|^2 = |v_j|^2$

suppose $U(gx) = U(x)$ pairwise interactions / central potential

Then g is a symmetry.

Suppose $g_r = \exp(r\mathbb{X})$ is a group of symmetries for some $\mathbb{X} \in so(d)$
Then $g' = \mathbb{X}$ so

$$g_r \cdot x_j = \exp(r\mathbb{X}) \cdot x_j$$
$$\text{so } \quad g'\big|_x = (\mathbb{X} x_j)_j$$

So the conserved quantity is $\sum_j \frac{\partial L}{\partial v_j} \cdot (g_r)_j = \sum_j m_j v_j \mathbb{X} x_j$

<u>Def:</u> Fix $x_0 \in \mathbb{E}^d$. The **angular momentum** of a particle of mass m at position x, moving at velocity $v \in \mathbb{R}^d$ is the linear functional $\underbrace{\quad}_{\text{on } \mathbb{E}^d}$ $L \in so(d)'$ given by

$$L(\mathbb{X}) = m\,(x - x_0)^* \mathbb{X} v.$$

Ex: Using basis $X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$, $X_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $X_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$,

check this is the usual definition

Cor: If U is invariant under rotation, total angular momentum is conserved

More linear alg: think of X as map $\mathbb{R}^n \to (\mathbb{R}^n)^*$ with dual map $X^*: (\mathbb{R}^n)^{**} = \mathbb{R}^n \to \mathbb{R}^{n*}$, $X^* = -X$.

so write (Xu, v) $u, v \in \mathbb{R}^n$

$(Xu, u) = (u, X^* u) = -(Xu, u) \Rightarrow (Xu, u) = 0$

Ex: let $u, v \in \mathbb{R}^d$. Then functional $x \mapsto u \mathbb{X} v$ depends only on plane spanned by u, v. (up to rescaling) if u, v independent conversely.

Pf: $(au + bv) \mathbb{X} (cu + dv) = ac\, \cancel{u \mathbb{X} u} + ad\, u \mathbb{X} v + bc\, v \mathbb{X} u + bd\, \cancel{v \mathbb{X} v} =$

$= (ad - bc)\, u \mathbb{X} v$

conversely let w be indep of u, v, say u^*, v^*, w^* corresponding elements of a dual basis, let $\mathbb{X} = u^* w^* - w^* u^*$

Then $u \mathbb{X} v = 0$ since $w^*(v) = u^*(v) = 0$

But $u \,\mathbb{X}\, w = 1$. $\blacksquare$

<u>Prop</u>: Given L, can find $2k \le d$ orthonormal vectors $\{(x_j, v_j)\}_{j=1}^{k}$ s.t. L is a linear combination of $(\mathbb{X} \mapsto x_j^T \mathbb{X} v_j)$.

<u>Key Historical example: central potential</u>

Single mass in $\mathbb{R}^d$: $L = \frac{1}{2} m v^2 - U(r)$, $r = |x|$

$\Rightarrow$ angular momentum is conserved

At any time either x, v proportional or not.

$\Rightarrow$ angular momentum $= 0$, movement is on line.

or not, then $\Rightarrow$ motion restricted to the plane spanned by x, v.

$\Rightarrow$ enough to understand 2d motion

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - U(r)$$

two conserved quantities : $\begin{cases} E = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + U(r) \\ L = m r^2 \dot{\theta} \end{cases}$

<u>Cor</u>: sign of $\dot{\theta}$ fixed, area swept by radius vector is.

$$\int_{t_0}^{t_1} r\,d\theta = \int_{t_0}^{t_1} r\dot{\theta}\,dt = \frac{L}{mr}(t_1 - t_0)$$

Combining the equations $E = \frac{1}{2}m\dot{r}^2 + \hat{U}(r)$

effective potential $\rightarrow$ $\hat{U}(r) = U(r) + \frac{L^2}{2mr^2}$.

integrate this to get $r = r(t)$

integrate $\dot{\theta} = \frac{L}{mr^2}$ to get $\theta(t)$.

Cor: If U blows up at 0 slower that $\frac{1}{r^2}$, can't approch origin