Math 428, lecture 12

Last time: Angular momentum

Infinitesimal rotation $X \in so(d) = \{X \in M_d(\mathbb{R}) \mid X^T + X = 0\}$
induces 1-param group $\{g_r = \exp(rX)\} \subset SO(d) \circlearrowright \mathbb{R}^d$.

If $L = \frac{1}{2} m v^2 + U(x)$ is symmetric by g_r the quantity
$m x^T X v$
is conserved.

If L is $SO(d)$-invariant then conserved quantity is
$\ell : so(d) \to \mathbb{R} : \quad \ell(X) = m x^T X v$.

HW: Define $[X, Y] = XY - YX$ ("commutator")
(1) If $X, Y \in so(d)$, so is $[X, Y]$.
(2) $g^Y_{-\varepsilon} \cdot g^X_{-\varepsilon} \cdot g^Y_\varepsilon \cdot g^X_\varepsilon \simeq g^{[X,Y]}_{\varepsilon^2}$

(3) If L is inv't g^X_r, g^Y_r also inv't by $g^{[X,Y]}_r$.
(4) can compute conserved quantities from set of generators.
(can't have $\sim d^2$ independent conserved quantities if ODE lives on space of dim $2d$)

Central potential: $L = \frac{1}{2}mv^2 - U(|x|)$

saw: each orbit confined to line / plane.

planar case: can use conservation of $\Big\}$ Energy / Angular momentum

to integrate system.

combination: $\epsilon = \frac{1}{2}m\dot{r}^2 + \tilde{U}(r)$; $\tilde{U}(r) = U(r) + \frac{L^2}{2mr^2}$

$\Rightarrow$ $\dot{r}^2$ fcn of r, at each radius two possibilities

$\Rightarrow$ either orbit unbounded (come from ∞, go back to ∞)

or bounded $r_{min} \le r \le r_{max}$

solutions to $U(r) + \frac{L^2}{2mr^2} = \epsilon$

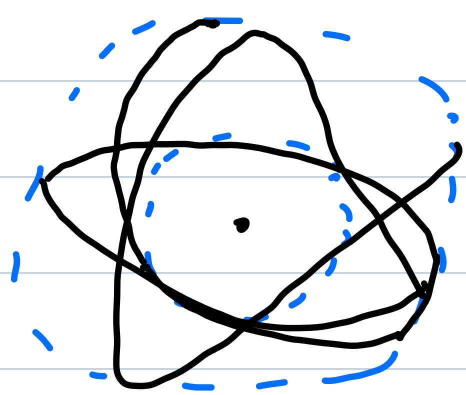
Angle change during period:

$$\Delta\theta = 2\int_{r_{min}}^{r_{max}} \dot{\theta}\,dt = 2\int_{r_{min}}^{r_{max}} \frac{\dot{\theta}}{\dot{r}}\,dr = \frac{2L}{\sqrt{2m}}\int_{r_{min}}^{r_{max}} \frac{dr}{r^2\sqrt{\epsilon - \tilde{U}(r)}}$$

observe: orbit is periodic iff $\Delta\theta \in \mathbb{Q}^{\times}\cdot\pi$

rational multiple of π

path can be:



(Fact: if $U(r) = r^{\alpha}$ get closed orbits iff $\alpha = -1$ or $\alpha = 2$)

Remarks In 1859 realized orbit of Mercury not closed, rotates by 1.5 degrees every 100 years

Solution: (Einstein) potential needs correction due to GR.

HW: two-body problem ⇒ 1-body problem

Small Oscillations

Knows if $L = \frac{1}{2}\langle M(x)v, v\rangle - U(x)$

⇒ EoM

$$\frac{d}{dt} M(x) v = -dU + \frac{1}{2}\langle dM\, v, v\rangle$$

⇒ if $dU(x_0) = 0$ then $x(t) \equiv x_0$ is a solution $(v(t) = 0)$

Get equilibrium / stationary state / fixed point

Let $q = x - x_0$. To first order in $q, \dot{q}$

$$M(x_0)\ddot{q} \approx -H(x_0) q$$

$H = d^2U$ Hessian of U

let $y = \sqrt{M}\, q$ $\quad \ddot{y} = -\tilde{H} y$

$$\tilde{H} = M^{-\frac{1}{2}} H M^{\frac{1}{2}}$$

(so find basis of $\mathbb{R}^q$, $\underline{r}_i$, $\quad H\underline{r}_i = \omega_i^2 M \underline{r}_i$

then $q(t) = \sum_{i=1}^{d} a_i(t) r_i$, $a_i(t) = A_i \sin(\omega_i t) + B_i \cos(\omega_i t)$

(30)

In detail, diagonalize $\hat{A}$, get basis $\underline{r}_i$, e.v. ω_i^2

Then use Ansatz (*)

$$M\ddot{q} = \sum_{i=1}^{d} \ddot{a}_i(t) M \underline{r}_i \quad , \quad -Hq(t) = -\sum_{i=1}^{d} a_i(t)\,\omega_i^2 M \underline{r}_i$$

so $M\ddot{q} = -Hq$ iff $\ddot{a}_i = \omega_i^2 a_i$.

(to find r_i , $H\underline{r} = \lambda M \underline{r}$ $\Leftrightarrow$

$$(M^{-1/2} H M^{-\frac{1}{2}})(M^{+\frac{1}{2}}\underline{r}) = \lambda (M^{\frac{1}{2}}\underline{r})$$

symmetric eigenvector

Call r_i the normal modes / charcteristic modes

ω_i the normal/characteristic frequencies

The system with Lagrangian $\frac{1}{2}\langle Mv, v\rangle - \frac{1}{2}\langle Kx, x\rangle$
is called "harmonic oscillator".

Can add friction ("damped harmonic oscillator")
can add forcing (periodic forcing) consider ODE:

$$M\ddot{q} = -Hq + F\cdot\cos(\omega t)$$

part oscillaters at frequency ω will have amplitude of the form $g(\omega)\cdot F$, g = "Frequency response curve"

Example:

q_j = displacement of j^{th} mass from equilibrium

so $U = \frac{1}{2}\sum_{j=1}^{N} k(q_j - q_{j-1})^2$ $q_0 = 0$, $q_{N+1} = 0$ pinned

$$\Rightarrow H = \begin{pmatrix} 1 & -1 & & & 0 \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ 0 & & & -1 & 1 \end{pmatrix}$$

Ex: diagonalize this H

QM case of asymptotics

$$-\frac{\hbar^2}{2m}\Delta\psi = E\psi \iff \Delta\psi = -\underbrace{\frac{2mE}{\hbar^2}}_{\lambda\to\infty}\psi$$

(Fact: $e^{i\frac{S}{\hbar}} \approx \psi$)