Math 428, Lecture 13

Last time: small oscillations:

near a potential minimum, equations of motion are

$$M(x_0)\ddot{q} \approx -H(x_0)\,q \qquad q = x - x_0$$

H = Hessian of U at x_0

$$\Rightarrow \quad \ddot{y} = -\tilde{H}y \quad \text{if } y = M^{1/2}q$$

$$\tilde{H} = M^{+\frac{1}{2}}HM^{-1/2}.$$

So $\tilde{H}$ has same eigenvalues as H, different eigenvectors

ω^2 ", ω = angular frequencies.

(can solve linear ODE via diagonalization, so nice to approximate by linear system).

Today: Rigid body motion

Def: A rigid body is a system of masses subject to the constraints that all pairwise distances are fixed

Assume: masses are in general position: not all in some proper affine subspace [of dim less than $d-1$]

Lemma: Let $A, B \subset \mathbb{E}^d$, $f: A \to B$ a distance-preserving bijection ($d(f(a), f(a')) = d(a, a')$)

Example: 

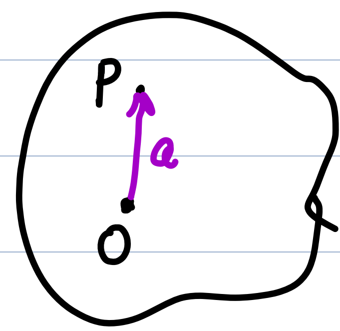
Then there exists an isometry $\tilde{f}: \mathbb{E}^d \to \mathbb{E}^d$ s.t. $f = \tilde{f}|_A$.
$\tilde{f}$ is unique if A is in general position

(#A = 3: SSS theorem for triangle congruence)

Cor: The configuration of a rigid body relative to a fixed = reference embedding is determined by an element of the Euclidean group

$\Rightarrow$ configuration space has dim $d + \frac{d(d-1)}{2} = \frac{d(d+1)}{2}$
(6 numbers if $d = 3$)

In reference embedding fix point O, parametrize point P by $\vec{a} = P - O$.



In general, we may rotate about O by $g \in SO(d)$ then translate O by x. The point P is then located at

$$X = x + ga. \leftarrow$$ in config (x, g), point P is at this location

Then point P is moving at velocity

$$V = \frac{dX}{dt} = \dot{x} + \dot{g}a$$

Here $v = \dot{x}$ is the velocity of O, $\dot{g} = T_g SO(d)$

<u>Def:</u> The angular velocity of the body is $\Omega = \dot{g}g^{-1}$.

so $\Omega \in T_1 SO(d) = so(d) = \{X : X + X^T = 0\}$

(<u>Proof:</u> consider $\frac{d}{dt}(g(t)g(t_0)^{-1})$, at time t_0)

$\Leftrightarrow \quad \dot{g} = \Omega g$

Then $$V = v + \Omega g a.$$

3d exercise

$d = 3$

<u>Thm:</u> (Euler) any $g \in SO(d)$ has a fixed vector

any $X \in so(d)$ has a null vector.

<u>Pf:</u> $g^* = g^{-1} \Rightarrow$ eigenvalues satisfy $\lambda^* = \lambda^{-1} \Leftrightarrow \lambda\lambda^* = 1$,

so lie on unit circle

g real $\Rightarrow$ e.v. come in conjugate pairs.

$\Rightarrow$ if $d = 3$ have eigenvalues $w, \bar{w}, 1$

(not -1 since $w\bar{w} \cdot 1 = 1$)

similarly if $X^* = -X$ ev. are iw, $w \in \mathbb{R}$, but $\operatorname{tr} X = 0$.

Cor: generic rotation is in a plane perpendicular to a well-defined axis,

generic infinitesimal rotation: same

Can write $\Omega = w_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} + w_2 \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ +1 & 0 & 0 \end{pmatrix} + w_3 \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

in general, get component of Ω for each pair of coordinate axes.

Lemma: Ω is independent of reference configuration

Pf: Say we rotate by h so P is initially at $ha = a'$

Then $X = x + g'a' = x + g'h^{-1}a$

so $g = g'h^{-1}$ so $\dot{g} = \dot{g}'h^{-1}$

$$\Omega = \dot{g}g^{-1} = \dot{g}'h^{-1}h(g')^{-1} = \dot{g}' \cdot (g')^{-1} = \Omega'.$$

Say we have many points, jth point at a_j, mass m_j.
Then the total kinetic energy is

$$\frac{1}{2}\sum_j m_j V_j^2 = \frac{1}{2}\sum_j m_j \langle v + \Omega g a_j, v + \Omega g a_j \rangle$$

$$= \frac{1}{2}\Big(\sum_j m_j\Big)v^2 + \langle v, \Omega g \sum_j m_j a_j \rangle + \frac{1}{2} m_j \langle \Omega g a_j, \Omega g a_j \rangle$$

First term is $\frac{1}{2}Mv^2$, $M = \sum_j m_j$

Second term: $\frac{\sum_j m_j a_j}{M} = \bar{a}$ is the location of the centre of mass relative to O.

get $\langle Mv, \Omega g \bar{a} \rangle = M \langle v, \Omega(\bar{x} - x) \rangle$

(1) choose O to be CM, term vanishes

(2) O fixed in space this is motion of CM about axis

Third term: $\frac{1}{2} \sum_j m_j \langle \Omega r_j, \Omega r_j \rangle$ $r_j = g a_j$

$$= \frac{1}{2} \sum_j m_j \left(r_j^T \Omega^T \Omega r_j \right) = \frac{1}{2} \sum_j m_j \operatorname{Tr}\left(r_j^T \Omega^T \Omega r_j \right)$$

$$= \frac{1}{2} \operatorname{Tr}\left(I \Omega^T \Omega \right), \quad I = \sum_j m_j r_j r_j^T$$

$$= \frac{1}{2} \operatorname{Tr}\left(\Omega I \Omega^T \right)$$

tensor of inertia

positive-definite symmetric matrix

Define $I_0 = \sum_j m_j a_j a_j^T$

Then the rotational term is

$$\frac{1}{2} \operatorname{Tr}\left(I \Omega^T \Omega \right) = \frac{1}{2} \operatorname{Tr}\left(g I_0 g^T \Omega^T \Omega \right)$$

$$= \frac{1}{2} \operatorname{Tr}\left(I_0 \, g^T \Omega^T \Omega g \right) = \frac{1}{2} \operatorname{Tr}\left(I_0 \, \dot{g}^T \dot{g} \right)$$

summary: $T = \frac{1}{2}Mv^2 + M\langle v, \Omega(\bar{x} - x)\rangle + \frac{1}{2}\mathrm{Tr}(I\Omega^T\Omega)$

CM translation — CM rotation about O — rotation about CM

Def: The eigenvectors of I_0 / I are axes of inertia

The angular momentum is then $J = \frac{\partial T}{\partial \dot{g}}$.

Place O at CM. Then J is as follows

$$\frac{1}{2}\mathrm{Tr}(I_0(\dot{g}^T + Y^T)(\dot{g} + Y)) = \frac{1}{2}\mathrm{Tr}(I_0 \dot{g}^T \dot{g})$$
$$+ \frac{1}{2}\mathrm{Tr}(I_0 \dot{g}^T Y + I_0 Y^T \dot{g}) + O(Y^2)$$

$$J : \text{map } Y \mapsto \frac{1}{2}\mathrm{Tr}(I_0 \dot{g} Y + Y^T \dot{g} I_0)$$
$$= \frac{1}{2}\mathrm{Tr}((I_0 \dot{g} + I_0 \dot{g}^T) Y)$$

$$\Rightarrow \quad J = \frac{1}{2}(I_0 \dot{g} + I_0 \dot{g}^T)$$

in terms of Ω: $\dot{g} = \Omega g$: $J = \frac{1}{2}(\Omega I + I\Omega)$

$$(I\Omega)^T = \Omega^T I^T = -\Omega I.$$

for general $Y \in so(d)$ $\mathrm{Tr}(I\Omega Y) = \mathrm{Tr}(JY)$

Example: Free body. J · constant:

Then $\frac{dJ}{dt} = I\dot{\Omega} + \Omega J$

if $\frac{dJ}{dt} = 0$, in basis of axes inertia set

$$\begin{cases} I_1\dot{\omega}_1 + (I_3 - I_2)\,\omega_2\omega_3 = 0 \\ I_2\dot{\omega}_2 + (I_1 - I_3)\,\omega_3\omega_1 = 0 \\ I_3\dot{\omega}_3 + (I_2 - I_1)\,\omega_1\omega_2 = 0 \end{cases}$$

initially, $\omega_1 = \omega_2 = 0$

say we make small perturbation, ω_1, ω_2 small, $\neq 0$
to 1st order, initially ω_3 remains same

Then

$$\dot{\omega}_1 = -\left(\frac{I_3 - I_2}{I_1}\,\omega_3\right)\omega_2$$

$$\dot{\omega}_2 = -\left(\frac{I_1 - I_3}{I_2}\,\omega_3\right)\omega_1$$

So $\begin{pmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \end{pmatrix} = \omega_3 \begin{pmatrix} 0 & -\frac{I_3 - I_2}{I_1} \\ \frac{I_3 - I_1}{I_2} & 0 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$

e.v. are $\pm i\omega_3 \sqrt{(I_3 - I_2)(I_3 - I_1)}\, \frac{1}{\sqrt{I_1 I_2}}$

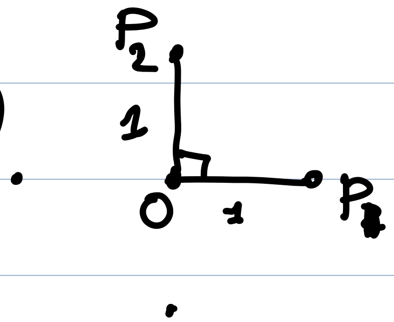
general position

need: affine hull of points has dim $\geq d-1$

for co-ords

need dim d for $I_0 > 0$

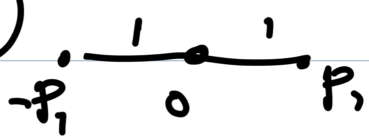
③d



$$I_0 = m(\vec{e}_1 \vec{e}_1^{\,T} + \vec{e}_2 \vec{e}_2^{\,T})$$

$$= \begin{pmatrix} m & & \\ & m & \\ & & 0 \end{pmatrix}$$

②d



$$g \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} g^{-1} \Omega^T \Omega$$

$$\Omega = w \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \qquad -\Omega^2$$

$$\Omega^T \Omega = +w^2 \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$$

$$w^2 \,\mathrm{Tr}\left(g \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} g^{-1}\right) = w^2$$

$$I_0 = \sum_j m_j a_j a_j^T$$

not interested.

$$v^T I_0 v$$

$$\mathrm{Tr}(\Omega^T I_0 \Omega)$$

$$I_0 = \begin{pmatrix} I_1 & & \\ & I_2 & \\ & & I_3 \end{pmatrix} \qquad \Omega = w_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} + w_2(\ ) + w_3(\ )$$

$$\mathrm{Tr}(g I_0 g^T \Omega^T \Omega) = \cdot\, w_1^2 + \cdot\, w_1 w_3 + \cdot\, w_3^2 =$$