Math 412, lecture 14

Last time: Rigid body motion

Today: Hamiltonian mechanics

(1) convexity & Legendre transformations
(2) Hamiltonians & Hamilton's equations

Physics $H = pv - L$ what does this mean?

(1) Convexity

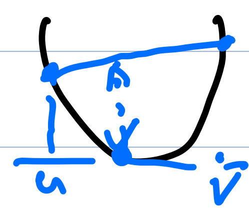
Fix a f.d. real vector space V

Def: A subset $C \subset V$ is **convex** if $\forall u, v \in C$, $t \in [0,1]$
$(1-t)u + tv \in C$.

A function $f : C \to \mathbb{R}$ defined on convex set is **convex** if $\forall u, v \in C$, $t \in [0,1]$

$$f((1-t)u + tv) \leq (1-t) f(u) + t f(v)$$

strictly convex if $<$ when $u \neq v$, $t \neq 0, 1$.

Geometrically: f convex  secant line is above graph

Ex: f is convex iff the epigraph $\{(u,a) \mid f(u) \le a\}$ is convex in $V \times \mathbb{R}$.

Remark: Allow f to take value ∞, then might as well have them defined on V, check: $\{u \mid f(u) < \infty\}$ is convex

Ex: Epigraph is closed iff f is lower semicontinuous $\forall v \, \forall \varepsilon > 0 \, \exists U$ nbd of v st. if $u \in U$,
$$f(u) \ge f(v) - \varepsilon$$

Ex: Let $f: I \to \mathbb{R}$ be convex, $I \subset \mathbb{R}$ interval
(1) slope of secant line from $(u, f(u))$ to $(v, f(v))$, is monotone in u, v.
(2) f has left & right derivatives at every point & if $u < v$
$$f_L'(u) \le f_R'(u) \le f_L'(v) \le f_R'(v)$$
(3) At most countably many points where f is non-diff, which are jump discontinuities, f' is monotone.

Cor: we can parametrize points u by slopes $f'(u)$ (if f is strictly convex, f' strictly increasing)

The mapping $u \mapsto \underset{df_u}{f'(u)}$ is a map $V \to V^*$

Def: The Fenchel conjugate or convex conjugate or Legendre transform of $f: V \to \mathbb{R} \cup \{\infty\}$ is the function f^* defined on V^* by

$$f^*(p) = \sup_{v \in V} \left[\langle p, v \rangle - f(v) \right]$$

(f is "proper" if $f(v) \neq \infty$ for some v)

Example: (1) $f(x) = \langle m, x \rangle + b$

$$\sup_{x \in V} \left(\langle p - m, x \rangle - b \right) = \begin{cases} -b & p = m \\ \infty & p \neq m \end{cases}$$

(2) For $1 < r < \infty$, if $f(x) = \frac{1}{r} |x|^r$, $f^*(p) = \frac{1}{r^*} |p|^{r^*}$

$$\frac{1}{r} + \frac{1}{r^*} = 1.$$

(3) Let $g: V \to V^*$ be symmetric, pos. def. and let $f(x) = \frac{1}{2} \langle gx, x \rangle + \langle m, x \rangle + b$ then

$$f^*(p) = \frac{1}{2} \langle p - m, g^{-1}(p - m) \rangle - b$$

Pf: Given p let $h = g^{-1}(p - m) \in V$.

Then $\langle p, x \rangle - f(x) = \langle p - m, x \rangle - \frac{1}{2} \langle gx, x \rangle - b$

completing square

$$= -\frac{1}{2} \langle g(x - h), (x - h) \rangle + \frac{1}{2} \langle gh, h \rangle - b$$

maximized at $x = h$ whence claim

$$\langle gh, h \rangle = \langle p - m, g^{-1}(p - m) \rangle$$

Observation: Fenchel's inequality $f(v) + f^*(p) \geq \langle p, v \rangle$

Lemma: let f be any function

(1) f^* is lsc & convex

(2) $\{(v,a) : f^{**}(v) \le a\} = \overline{\mathrm{Conv}}(\{(v,a) : f(v) \le a\})$

closed convex hull

Theorem: (Fenchel-Moreau) i.: f is proper, convex, "closed" (lsc) then $f^{**} = f$.

Key point: K convex, closed, $x \notin K$ $\exists p$ s.t. $p(k) \le \lambda < p(x)$

If f is strictly convex, diff, choose $u \in \mathrm{dom}(f)$

consider $f^*(df_u)$: function $v \to \langle df_u, v\rangle - f(v)$

has its unique critical point at u

$$d[\langle df_u, v\rangle - f(v)] = df_u - df_v$$

$$\Rightarrow \quad f^*(df_u) = \langle df_u, u\rangle - f(u)$$

Phase space & the Hamiltonian

Def: Phase space or the cotangent bundle is the space

$$T^*X = \{(x,p) \mid \begin{matrix} x \in X \\ p \in (T_x X)^* \end{matrix}\}$$

Now fix a Lagrangian $L: TX \times E^1 \to \mathbb{R}$

Assume $v \mapsto L(x,v;t)$ is strictly convex on $T_x X$ for x,t fixed

<u>Def:</u> The Hamiltonian of the system is the function

$H: T^*X \times E^1 \to \mathbb{R}$

s.t. for fixed x,t, $p \to H(x,p;t)$ is the convex conjugate of $v \to L(x,v;t)$

<u>Example:</u> $L(x,v) = \frac{1}{2}\langle M(x)v, v\rangle - U(x)$

then $H(x,p) = \frac{1}{2}\langle M^{-1}(x)p, p\rangle + U(x)$

(if only dependence on v is in quadratic kinetic term, then $H = T + U$)

To think on $\frac{1}{2}\langle Mv, v\rangle$, $\frac{1}{2}\langle M^{-1}p, p\rangle$ as "same function", identify state of motion (x,v,t) with phase point (x,p,t) where $p = \frac{\partial L}{\partial v}\Big|_{(x,v,t)}$

<u>Observe:</u> This depends on <u>choice</u> of L

Aside: in optimization, have function f on V (say f is convex) want to minimize f.

<u>Paradigm</u>: start at point $x_0 \in V$ step down against gradient ∇f, in cts time have ODE $\dot{x} = -\nabla f(x)$

then $\frac{d}{dt} f(x) = -\langle \nabla f, \nabla f \rangle \leq 0$

<u>But</u> the derivative of f is $df_x \in V^\vee$

need a rule mapping $V^\vee \to V$ (i.e. a metric / inner product)

Still for quadratic Lagrangian,

$$L = \frac{1}{2}\langle Mv, v\rangle - U(x)$$

$$p = \frac{\partial L}{\partial v} = Mv \quad , \quad v = M^{-1}p$$

Then $H = pv - L$

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§2 <u>The Hamiltonian flow</u>

The physical path $\gamma(t) = x(t)$ is a path of states of motion $(x(t), v(t) : t)$ $\quad v(t) = \dot{x}(t)$

$\Rightarrow$ path in phase space (x, p, t)

To start with $\dot{x} = v = \frac{\partial H}{\partial p}$

what about $\dot{p}$? $\quad \dot{p} = \frac{\partial L}{\partial x}$ $\quad$ (Euler-Lagrange)

$L = pv - H$ so maybe $\frac{\partial L}{\partial x} = -\frac{\partial H}{\partial x}$?

This is nonsense: $\frac{\partial L}{\partial x} = \left(\frac{\partial L}{\partial x}\right)_v$.

natural $\frac{\partial H}{\partial x}$ is $\left(\frac{\partial H}{\partial x}\right)_p$.

Prove of this nonsense

Think of H as function on TX

$$dH = d(pv - L) = \langle dp, v\rangle + \langle p, dv\rangle \underbrace{- \frac{\partial L}{\partial v} dv}_{-p\,dv} - \frac{\partial L}{\partial x} dx - \frac{\partial L}{\partial t} dt$$

$$= \langle dp, v\rangle - \frac{\partial L}{\partial x} dx - \frac{\partial L}{\partial t} dt$$

$$\Rightarrow \quad v = \frac{\partial H}{\partial p}, \quad -\left(\frac{\partial L}{\partial x}\right)_v = \left(\frac{\partial H}{\partial x}\right)_p, \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

Now push everything to T^*X

$$\Rightarrow \begin{cases} \dot{x} = \frac{\partial H}{\partial p} \\ \dot{p} = -\frac{\partial H}{\partial x} \end{cases} \Leftarrow \text{Hamilton's equations}$$

$\Rightarrow$ $z = (x, p)$ $\quad \dot{z} = X_H$ $\quad X_H$ = vector field on T^*X

If have $dF = a\,dx + b\,dy$

if $v = \begin{pmatrix} v_x \\ 0 \end{pmatrix}$ then $\langle dF, v\rangle = a \cdot v_x$

i.e. $a = \left(\frac{\partial F}{\partial x}\right)_y$

UBC library has online copy of Landau & Lifshitz.