Math 928, lecture 15

Last time: convex analysis → legendre transform:

f on V is convex,

$$\boxed{f^*(p) = \sup_{v \in V} \langle p, v \rangle - f(v)}$$

If f is diff, at each v the sup is at p where $p = \frac{df}{dv}$, then

$$f^*(p) = pv - f(v)$$

Example: $L = \frac{1}{2}\langle Mv, v\rangle - U(x)$ fcn on TX

$\Downarrow$

$$H = \frac{1}{2}\langle M^{-1}p, p\rangle + U(x) \quad \text{fcn on } T^*X$$

phase space

$\Downarrow$

Equations of motion in T^*X are

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial p} \\ \dot{p} = -\frac{\partial H}{\partial x} \end{cases}$$

Hamilton's equations.

Today: ① examples
② symplectic structure

Example: Harmonic oscillator

$$L = \frac{1}{2} m v^2 - \frac{1}{2} k x^2 \Rightarrow H = \frac{1}{2m} p^2 + \frac{1}{2} k x^2$$

$$\Rightarrow \dot{x} = \frac{p}{m}, \quad \dot{p} = -kx$$

$$\Leftrightarrow \frac{d}{dt}\begin{pmatrix} x \\ p \end{pmatrix} = \begin{pmatrix} & 1/m \\ -k & \end{pmatrix}\begin{pmatrix} x \\ p \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ p \end{pmatrix} = \exp\left\{\begin{pmatrix} & 1/m \\ -k & \end{pmatrix} t\right\} \cdot \begin{pmatrix} x \\ p \end{pmatrix}.$$

(the path $\begin{pmatrix} x(t) \\ p(t) \end{pmatrix}$ traces ellipses in T^*X,

these are the level sets $H = \text{const}$)

Example: Central potential

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - U(r)$$

$$\Rightarrow p_r = m\dot{r}, \quad J = p_\theta = m r^2 \dot{\theta}$$

$\Rightarrow$

$$H = p_r \dot{r} + p_\theta \dot{\theta} - L = \frac{p_r^2}{2m} + \frac{J^2}{2mr^2} + U(r)$$

$$\Rightarrow \begin{cases} \dot{r} = \dfrac{p_r}{m} \\ \dot{p}_r = -\dfrac{dU}{dr} - \dfrac{J^2}{mr^3} \\ \dot{\theta} = \dfrac{J}{mr^2} \\ \dot{J} = 0 \quad \text{since } H \text{ indep of } \theta. \end{cases}$$

$$\boxed{\begin{aligned} \dot{x} &= \frac{\partial H}{\partial p} \\ \dot{p} &= -\frac{\partial H}{\partial x} \end{aligned}}$$

As before J conserved, can isolate $r(t)$ and recover $\theta(t)$.

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(2) Hamiltonian flow & the symplectic structure.

Hamilton's equations are first-order, so they have form

$$\begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = X_H,$$

X_H is a vector field on T^*X

$[X_H(x,p) \in T_{(x,p)}(T^*X).]$

How did we get the vector field from differential dH?

Have linear map $\omega_{(x,p)} : T_{(x,p)} T^*X \to T^*_{(x,p)}(T^*X)$

and Hamilton's equations are

$$\frac{d}{dt}\begin{pmatrix} x \\ p \end{pmatrix} = \omega^{-1}(dH)$$

in basis coming from coord $\begin{pmatrix} x \\ p \end{pmatrix}$,
ω has matrix

$$J = \begin{pmatrix} & I \\ -I & \end{pmatrix}$$

(if we use basis $\begin{bmatrix} x_1 \\ p_1 \\ x_2 \\ p_2 \\ \vdots \end{bmatrix}$, $J = \begin{pmatrix} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} & & \\ & \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} & \\ & & \ddots \end{pmatrix}$)

Def: An observable is a smooth function

$$A \in C^\infty(\underbrace{T^*X}_{\text{phase space}} \times \underbrace{\mathbb{R}}_{\text{time}})$$

Then

$$\frac{dA}{dt} = \frac{\partial A}{\partial x}\cdot\dot{x} + \frac{\partial A}{\partial p}\dot{p} + \frac{\partial A}{\partial t}$$

$$= \frac{\partial A}{\partial x}\frac{\partial H}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial H}{\partial x} + \frac{\partial A}{\partial t}$$

$dA\cdot J^{-1}dH$

$$\omega(dA, dH) + \frac{\partial A}{\partial t}.$$

Note: ω is an antisymmetric bilinear form on $TJ^1\mathbb{X} \cong J^0J^1\mathbb{X}$

(i.e. a 2-form)

we will use a bit of the theory of differential forms

In co-ords $\binom{x}{p}$, ω is constant, so $d\omega = 0$ (ω is closed).

Def: A symplectic manifold is a pair (M,ω) where M is a smooth manifold, ω is a non-degenerate closed 2-form

Worry: maybe ω depended on choice of co-ords?

Calculation: X is any manifold (config space)

$M = T^*X$, $\pi : T^*X \to X$ canonical projection

$\pi(x,p) = x.$

Then

$d\pi_{(x,p)} \in \mathrm{Hom}_{\mathbb{R}}(T_{(x,p)}M, T_x X)$

$\Rightarrow$ get map $d\pi^*_{(x,p)} : T^*_x X \to T^*_{(x,p)} T^*X.$

So $\Theta_{(x,p)} \overset{\text{def}}{=} d\pi^*_{(x,p)}(p) \in T^*_{(x,p)} T^*X$

i.e. $\Theta_{(x,p)}$ is a 1-form on $M = T^*X$.

Check: $\omega = -d\Theta$.

Def: A co-ordinate system $\{q_i, p_i\}_{i=1}^n \subset M$ is canonical if

$$\omega = \sum_{i=1}^n dq_i \wedge dp_i$$

$\Leftrightarrow$ matrix of ω is J.

$M = T^*X$

Example: $\{q_i\}$ any co-ords on X

$\{p_i\}$ conjugate momenta wrt L.

Theorem: (Darboux) A symplectic manifold always has canonical co-ordinate systems

Defn: The Poisson bracket of $A, B \in C^\infty(M)$ is the observable

$$\{A,B\} = \omega(dB, dA) = -\omega(dA, dB)$$

The Hamiltonian vector field associated to an observable H is

$$X_H = \omega^{-1}(dH)$$

In this language, Hamilton's equations are

$$\boxed{\dot{z} = X_H(z)} \qquad (z(t) \in M = T^*Q)$$

wrt to this flow, $\dot{A} = \{H, A\} + \frac{\partial A}{\partial t}$.

In particular (if H, A time-indep), A is a constant of motion = conserved quantity iff $\{H, A\} = 0$ ("A, H Poisson-commute")

Note: Nothing special about H, can be any observable. Checks flows defined by H, A commute iff $\{H, A\} = 0$ (flow X_A is a symmetry)

Def: A (smooth) map $F: M \to M$ preserving the symplectic structure is called a symplectomorphism or a canonical transformation.

(if $\omega \in \Omega^2 M$, $F: N \to M$, define $F^* \omega$ by

$$(F^*\omega)(X, Y) = \omega_{F(p)}(dF(X), dF(Y))$$

for $X, Y \in T_p N$)

Prop: The Hamiltonian flow is a symplectomorphism

(define for $z \in T^*X$, $F_{s,t}(z) = \begin{pmatrix} x(t) \\ p(t) \end{pmatrix}$ if $\begin{pmatrix} x(s) \\ p(s) \end{pmatrix} = z$

= flow generated by X_H)

Note: $F_{t,r} \circ F_{s,t} = F_{s,r}$

If H time-indep, $F_{s,t} = F_{t-s}$, F_t is a group (1-param)

<u>Proof:</u> direct calculation

$$\frac{d}{dt}\left(F_{s,t}^{*} \omega\right) = F_{s,t}^{*}\left(\mathcal{L}_{X_H} \omega\right)$$ composition group + eqn of motion

$$= F_{s,t}^{*}\left(i_{X_H} d\omega + d\left(i_{X_H}\omega\right)\right)$$ Cartan's formula

$$= F_{s,t}^{*}\left(0 + d(dH)\right)$$ defn of X_H

$d^2 = 0$

$$= 0$$

Liouville's Theorem

<u>Corollary:</u> The flow preserves the <u>volume</u> <u>form</u> assoc to ω

In canonical co-ords assoc to box $dq_1 \ldots dq_n dp_1 \ldots dp_n$ "volume" 1. (if F is canonical, dF preserves J, i.e.

$$dF^{T} \circ J \circ dF = J$$

so $\det(dF)^2 = 1$, so $|\det(dF)| = 1$.

Cor: Define $\Omega_\omega \in \Omega^{2n}(M)$ by
$\omega \wedge \omega \wedge \dots \wedge \omega$
(non-zero since ω is non-degen)

check: $\omega \wedge \omega \dots \wedge \omega = dq_1 \wedge dq_2 \dots \wedge dq_n \wedge dp_1 \wedge \dots \wedge dp_n$
$= dq_1 \wedge dp_1 \wedge dq_2 \wedge dp_2 \wedge \dots \neq 0$

This is the beginning of Ergodic Theory, the theory of measure-preserving tranformations

Another calculation:

$$\frac{d}{dt}\{A,B\} = \left\{\frac{dA}{dt}, \frac{dB}{dt}\right\} \text{ along } H$$

$$\Rightarrow \{H,\{A,B\}\} = \{\{H,A\},B\} + \{A,\{H,B\}\}$$

$$\Rightarrow \{\{A,B\},C\} + \{\{C,A\},B\} + \{\{B,C\},A\} = 0$$

Jacobi identity

$\Rightarrow \{\cdot,\cdot\}$ makes $C^\infty(T^*X)$ into a <u>Lie algebra</u>.

<u>Cor</u>: if H, A time-indep, $\{H, A\} = 0$

F_s^A commutes with F_t^H.

(diff wrt s and t $\quad F_s^A F_t^H z$)

$\Rightarrow$ If $z(t) = F_t^H(z)$ is a physical path,

so is $F_s^A z(t) = F_s^A F_t^H z = F_t^H(F_s^A z)$

(Aside: if $f: X \to X$ diffeo, lift to T^*X is canonical ("point transformation") we get here <u>more</u> potential symmetries.

(Think of F_t^A as a time-dependent co-ord change, we can use z as the co-ord, then eqn of motion is $\dot{z} = 0$)

(Jacobi identity: check instead that

$$X_{\{A,B\}} = [X_A, X_B]$$)

Poisson bracket: on (M,ω), $\forall\, A,B \in C^\infty(M)$
set

$$\{A,B\} = -\omega(dA, dB) \in C^\infty(M)$$

Lie bracket: $D(C^\infty(M)) = \{ X : C^\infty(M) \to C^\infty(M)$ linear s.t. $X(fg) = Xf \cdot g + f \cdot Xg \}$

derivations

(Ex: in co-ords $X = \sum_{i=1}^{d} a_i(x) \frac{\partial}{\partial x_i}$)

Fact: $[X,Y] = XY - YX \in D(C^\infty(M))$

Prop: On (M,ω), $X_{\{A,B\}} = [X_A, X_B]$

Hamiltonian v.f. of $\{A,B\}$