Math 428, Lecture 16

Last time: Hamiltonian flow

(M,ω) symplectic $\quad M = \{(x,p) : \begin{matrix} x \in X \\ p \in T_x^* X \end{matrix}\}$

$\omega = J = \begin{pmatrix} & I \\ -I & \end{pmatrix} \qquad \omega\left(\begin{pmatrix} \Delta x_1 \\ \Delta p_1 \end{pmatrix}, \begin{pmatrix} \Delta x_2 \\ \Delta p_2 \end{pmatrix}\right)$
$= (\Delta p_2, \Delta x_1) - (\Delta p_1, \Delta x_2).$

If $H \in C^\infty(M \times \mathbb{R})$ ("observable") get ODE

$\dot{z} = \omega^{-1}(dH_z) \leftarrow$ Hamiltonian vector field X_H

set

$F_{s,t}(z) = z(t)$ if $z(s) = z$.

In autonomous case (H indep of t), set one-parameter group F_t.

Key fact: F_t are symplectomorphism:

$F_t^* \omega = \omega$. $\qquad$ Cor: F_t preserve volume

(Also obtained Jacobi identity for Poisson brackets)

$\Rightarrow$ If $\{H, A\} = \{H, B\} = 0$

Then $\{H, \{A, B\}\} = 0$

so $\{A, B\}$ also constant of the motion

$\{H, A\} = X_H \cdot A$

Today: Introduction to dynamics

(1) Topological dynamics
(2) Ergodic theory

Poincaré. studying 3-body problem

Earlier: solved one-body problem in a central potential, saw this solved 2-body problem with central interaction.

1-body

in 2-body problem, either have bounded or escaping orbits $U_{ij} = \frac{m_i m_j}{r_{ij}}$

(inverse square law). orbits are conic sections

Paradigm in mechanics: integrability

if $\dim X = n$, $\dim M = T^*X = 2n$,

have $2n-1$ conserved quantities
$\underbrace{}_{\text{independent}}$

r^2, $\frac{1}{r}$

Example: central potential in 2d, conserved. (1) energy, (2) angular momentum, (3) ...

Poincaré: 3-body problem not integrable

(if $u(r) \neq r^2, \frac{1}{r}$, then orbits precess

(as we complete period of r-variable, θ doesn't move by rational angle).

$\Rightarrow$ closure of the set $\{z(t) : t \in \mathbb{R}\} \subset T^* \mathbb{X}$
is not ld

(on $\mathbb{X}$ closure is an annulus)

Assumption: H time-indep
later
level sets $H^{-1}(E)$ compact / bounded

Have phase space M, group $(F_t)_t : M \to M$

(can also restrict to $\mathbb{Z}$, look at $F_1 : M \to M$

Topological dynamics studies ~~a~~ continuous
map $f : M \to M$. (M metric space).

dynamics: have $f : M \to M$ we are interested
"long term behaviour", e.g. orbits $z, f(z), f(f(z)), \dots$

Def: Call $z \in M$ "wandering" if have nbd U of z s.t. eventually U doesn't come back:

$\exists N \; \forall n > N \quad f^{(n)}(U) \cap U = \emptyset$

(note: wandering set is open, f-inv't. complement, non-wandering set)

Assume level sets of H are compact.
Saw: (Liouville's thm) F_t is volume-preserving
Ω_ω volume form assoc to ω.

Can restrict the volume form to the energy surface. $\Omega_\omega \in \Lambda^n T^*(T^*X)$

Given $2n-1$ tangent vectors in $T\binom{\text{energy}}{\text{surface}}$

add a vector in perpendicular direction: ∇H.

Thm: If I have a finite measure space $A = \{ z \mid E_1 \leq H(z) \leq E_2 \}$, μ measure on A

function $H : A \to \mathbb{R}$

then $(H_* \mu)(f) = \mu(f \circ H)$
is a measure on image

(prob. energy is between $[a, b]$
is the phase volume of the region

$\{ z \mid a \leq H(z) \leq b \}$.)

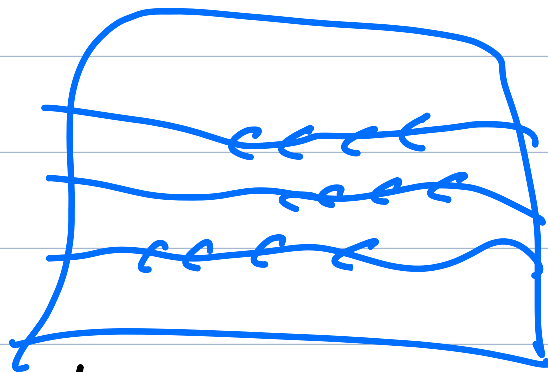
There exist unique prob measures μ_e for $e \in \mathbb{R}$, supported on $H^{-1}(e)$ s.t.

$$\int_A f \, d\mu = \int_{E_1}^{E_2} dH_*\mu(e) \int_{H^{-1}(e)} f \, d\mu_e$$

Now study dynamics on single energy surface $H^{-1}(E)$, with associated volume μ_E.

Liouville: Ω_ω is F_t-invariant

$\Rightarrow$ μ_ϵ are invariant.



Theorem: (Poincaré)
Let U be any set of positive measure (say open set) Then almost every point of U returns to U infinitely often.

Ergodic theory: study of measure/volume preserving actions

Pf: let $E \subset U$ be the points that "never come back"

$$E = \{ z \in U \mid f(z) \notin U, \ f(f(z)) \notin U, \dots \}$$

$$E = U \cap \bigcap_{n=1}^{\infty} f^{(-n)}(U).$$

$\forall z \in E$, $f(z) \notin E$.

sets $f^{(-k)}(E)$ all disjoint.

if $\quad f^{(k)}(z) \in E, \; f^{(\ell)}(z) \in E \quad k > \ell$

$f^{(\ell)}(z) \in E$

$f^{(k-\ell)}(f^{(\ell)}(z)) \in U$

contradiction

Have sets $\{ f^{(-k)}(E) \}_{k=0}^{\infty}$. All disjoint, all have same volume.

so $\mu_E(\cup \text{ sets}) = \sum_k \mu_E(f^{(-k)}(E))$

$$= \sum_{k=0}^{\infty} \mu_E(E)$$

since we assumed accessible phase space has finite volume, $\mu_E(E)$ has to be 0.

Define $\quad E_m = \{ z \in U \,/\, f^{(n)}(z) \notin U \text{ if } n > m \text{, } f^{(m)}(z) \in U \}$

then if $z \in E_m$, $f^{(m)}(z) \in E$

so $E_m \subseteq f^{(-m)}(E)$ so $\mu_E(E_m) \leq \mu_E(f^{(-m)}(E))$

$= \mu_E(E) = 0$

then $\mu_G\left(\bigcup_{m\geq 0} E_m\right) \leq \sum_{m=0}^{\infty} \mu_G(E_m) = 0$ ∎

Warning: no promise about when return happens

On "chaos":

In a completely integrable system if I move from z to z' close to z, all constants of motion are about same, orbits stay together forever

But in other systems typically arbitrarily small changes can cause solution to diverge (when this happens exponentially say the flow is Anosov)

See eg. Lorentz 1963

(in 2d see Poincaré - Bendixon)

Statisical mechanics studies "ensembles" on phase space.

Example: $G = SL_2(\mathbb{R}) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1 \}$

$$X = \mathbb{H}^2 = \{ x+iy \mid y>0 \}$$

$\Gamma = SL_2(\mathbb{Z}) =$

$$\Gamma \backslash G \simeq T_1^* X = \text{Energy surface}$$

Have geodesic flow = $\begin{pmatrix} e^{t/2} & \\ & e^{-t/2} \end{pmatrix}$

(also have unipotent flow $\begin{pmatrix} 1 & s \\ & 1 \end{pmatrix}$)