Math 428, Lecture 17

PS4 is posted, about symplectic integrators

Recently: Hamiltonian flow $\begin{cases} \dot{x} = \frac{\partial H}{\partial p} \\ \dot{p} = -\frac{\partial H}{\partial x} \end{cases}$

Better: $\begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = J^{-1} dH_{\binom{x}{p}}$ $\quad J = \begin{pmatrix} & I \\ -I & \end{pmatrix}$ $\quad$ (on $\mathbb{R}^n \oplus (\mathbb{R}^n)^*$)

Even better: $\dot{z} = \omega^{-1}(dH_z)$, $\quad z \in M = T^*X$

ω symplectic form on $T_z M$

Thm: Hamiltonian flow $\Phi_t : M \to M$ preserves ω, i.e. is a symplectomorphism.

HW: numerical schemes which act by symplectomorphisms are better. Preserve shadow Hamiltonian $\tilde{H}$ close to H

Touches on KAM theory (Kolmogorov-Arnold-Moser): if have an integrable system (dim $M = 2n$, have n conserved quantities), typical orbit lives on nondegenerate torus

KAM theorem: if $\tilde{H}$ close to H, still integrable, invariant torus is close

Last time: Since Φ_t preserves ω, it preserves volume, also on invariant submanifolds s.t. $H^{-1}(E)$ = energy surface

$\Rightarrow$ Poincaré recurrence: If $H^{-1}(E)$ has finite volume (e.g. compact) then if $U \subset H^{-1}(E)$ has positive volume, almost every $x \in U$ returns to U infinitely often.

Examples: If X cpt, then quadratic kinetic term ensures that $H^{-1}(E)$ is cpt in $N = T^*X$.

Today: connections to statistical mechanics, thermo.

Def: a measure on M is a rule assigning volumes to subsets of M $\Leftrightarrow$ rule allowing us to integrate cts fcns of compact support: $\mu(f) = \int_M f d\mu$.

Say μ is invariant if $\mu(\Phi_t^{-1}(U)) = \mu(U)$

$$\Leftrightarrow \mu(f \circ \Phi_t) = \mu(f).$$

In physics this is called an Ensemble

Example: microcanonical ensemble is the restriction of Ω_ω to $H^{-1}(E)$.

Example: canonical ensemble is the ensemble with density $\exp(-\beta H)$ wrt Ω_ω.

Thermodynamics: in practise n is very large. We use the ensemble to handle our ignorance of the state of the system

Often system is chaotic: over time, the trajectory in phase space is "random"

Theorem: TFAE for an ensemble ρ of mass 1.

(1) if U is an invariant set $\rho(U) \in \{0,1\}$.

(2) for almost every $z \in$ ensemble, for all $f \in C_c(M)$

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(\Phi_t(z))\,dt = \int_M f\,d\rho$$

"time average = space average"

Def: An ensemble of this type is called ergodic.

Theorem: Hard spheres in a box satisfy this (Sinai)

Actually, "ergodic hypothesis" is better formalized as mixing roughly: if z, z' two starting points then $(\Phi_t(z), \Phi_t(z'))$ look like two independent samples from ensemble.

Symmetries in Hamiltonian mechanics

$M = T^*X$ Phase space, H = Hamiltonian

say $A \in C^\infty(M)$, $\{H, A\} = 0$

let Φ_t = Hamiltonian flow for H. saw: $A \circ \Phi_t = A$.

$\Rightarrow$ if set R_s = flow of A, then $H \circ R_s = H$

more: 1-param groups Φ_t, R_s commute:

$$R_s(\Phi_t(z)) = \Phi_t(R_s(z))$$

$\Rightarrow$ R_s is a symmetry: maps solutions to solutions

New: R_s does not have to arise from function $X \to X$.