Math 428, lecture 18

Today: Integrable systems, Hamilton-Jacobi.

Say $\dim X = n$, so $\dim M = 2n$, $M = T^*X = \{(x,p)\}$

Def: A Hamiltonian $H \in C^\infty(M)$ is *integrable* if there are $n-1$ further constants of the motion $\{H_i\}_{i=2}^n$, $H_1 = H$ so that $\{H_i, H_j\} = 0$
and so that on a level set $M_{\underline{a}} = \underline{H}^{-1}(\underline{a})$ the vector fields $X_i = X_{H_i}$ are indep
$\underline{H} : M \to \mathbb{R}^n$ is the vector $(H_1, \ldots, H_n)$

- Since H_i are constants, $M_{\underline{a}} \subset M$ is invariant by flow (in fact by joint flows of all H_i)

- By implicit function thm, $(dH_1, \ldots, dH_n$ linearly indep$)$ the level set $M_{\underline{a}}$ is a submanifold (smooth)

- $\langle dH_i, \omega^{-1}(dH_j) \rangle = \langle dH_i, X_j \rangle =$ derivative of H_i along flow of X_j $= 0$,

$\Leftrightarrow \omega(X_i, X_j) = 0$ for all i, j.

$\Leftrightarrow$ restriction of ω to $TM_{\underline{a}}$ is identically 0

Def: Such n-dim submanifolds are called Lagrangian

Assumption: $M_{\underline{a}}$ is compact (e.g. because H_1 is proper)

Prop: Each connected component of $M_{\underline{a}}$ is a torus (diffeomorphic to $T^n = (\mathbb{R}/\mathbb{Z})^n = \mathbb{R}^n/\mathbb{Z}^n$)

Proof: let $\Phi^{(i)}_{t_i}$ be the 1-param group generated by H_i

these commute: $\Phi^{(i)}_{t_i}(\Phi^{(j)}_{t_j}(z)) = \Phi^{(j)}_{t_j}(\Phi^{(i)}_{t_i}(z))$

$\Rightarrow$ if we define $\Phi_{\underline{t}}(z) = \Phi^{(1)}_{t_1}(\Phi^{(2)}_{t_2}(\cdots(\Phi^{(n)}_{t_n}(z))\cdots))$
then

$$\Phi_{\underline{t}+\underline{s}} = \Phi_{\underline{t}} \circ \Phi_{\underline{s}} \qquad \underline{t} \in \mathbb{R}^n$$

("n-param group")

Orbits of this n-param group are n-dim, because at each point $d\Phi = (X_1, \ldots, X_n)$ which is of full rank
So they are open in $M_{\underline{a}}$ (map $\underline{t} \mapsto \Phi_{\underline{t}}(z)$ is invertible

hence open)

Two orbits are either equal or disjoint:
if $w = \Phi_{\underline{t}}(z) = \Phi_{\underline{s}}(z')$ then $z' = \Phi_{\underline{t}-\underline{s}}(z)$
so $\Phi_{\underline{u}}(z') = \Phi_{\underline{u}+\underline{t}-\underline{s}}(z)$.

So $M_q = \coprod$ orbits of $(\Phi_{\underline{t}})_{\underline{t} \in \mathbb{R}^n}$

so each orbit is closed, & is a connected component

By assumption, each orbit is compact

By orbit-stabilizer theorem, orbit is $\mathbb{R}^n / \Lambda$
$\Lambda = \{\underline{t} \mid \Phi_{\underline{t}}(z) = z\}$ if $\underline{\lambda} \in \Lambda$,

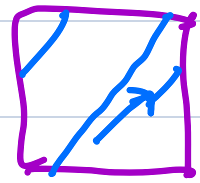
$$\Phi_{\underline{t}+\underline{\lambda}}(z) = \Phi_{\underline{t}} \Phi_{\underline{\lambda}}(z) = \Phi_{\underline{t}}(z)$$

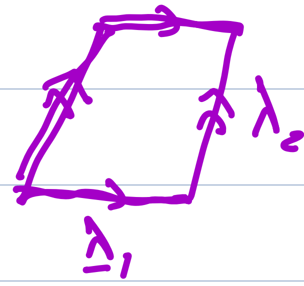
Now Λ is discrete (map $\underline{t} \mapsto \Phi_{\underline{t}}(z)$ is open)
so $\Lambda = \{ \sum_{j=1}^{J} n_j \underline{\lambda}_j \mid n_j \in \mathbb{Z} \}$, $\{\underline{\lambda}_j\}_{j=1}^{J} \subset \mathbb{R}^n$ linearly indep

Quotient is compact $\iff$ $J = n$, $\Lambda \cong \mathbb{Z}^n$, $\mathbb{R}^n / \Lambda$ = torus. ∎

Conclusion: If M_q is cpt, motion is always periodic (not necc. of each H_i).

Example: $n=2$





flow of H_1, H_2 can be in an irrational slope
But we can "rectify" things by changing direction

Idea: there exist observables whose flows are in the periodic directions. (functions of H_i)

$\Rightarrow$ change co-ords to "action-angle variables"
where co-ords θ_i are angles around torus
momenta I_i generate flow in θ_i direction.

Fix basis $\{\underline{\lambda}_j\}_{j=1}^n \subset \Lambda$ let $\{\gamma_j\}_{j=1}^n$ be corresponding closed curves in $M_{\underline{a}}$ (in torus we're working on)

Define $I_j = \oint_{\gamma_j} p\,dq = \sum_k \oint_{\gamma_j} p_k\,dq_k$

p_k = momentum corresp to q_k
$\{q_k\}$ co-ords on X
$p_k = \frac{\partial L}{\partial \dot{q}_k}$

Stokes's theorem: if $\gamma_j' - \gamma_j = \partial A$
$A \subset$ torus is a 2-cycle.
then

$$\oint_{\gamma_j' - \gamma_j} p\,dq = \oint_A dp \wedge dq = \oint_A \omega = 0$$ (ω is $\equiv 0$ on $M_{\underline{a}}$)

⇒ if we deform γ_j on torus, I_j does not change
so I_j is a constant on torus

$$\left(I_j(z) = \int_0^{\Delta_j} \Phi^*_{t,z}(p\,dq)\,dt \right)$$

if we move q, the tori deform continuously

<u>check:</u> angle θ_j on torus is conjugate to I_j.
<u>Better:</u> let $\{\theta_j\}$ be the conjugate coords to I_j
I_j constants of motion so $\frac{\partial H}{\partial \theta_j} = 0$

$$\Rightarrow H = H(I_1, \ldots, I_n) \Rightarrow \frac{d\theta_j}{dt} = \frac{\partial H}{\partial I_j} \text{ is constant.}$$

<u>Theorem:</u> (Kolmogorov - Arnol'd - Moser) Suppose the frequency vector $\omega_j = \frac{\partial \theta_j}{\partial t}$ is sufficiently irrational

then if we move H a little, the new $\tilde{H}$ is still integrable, torus moves contiously.

Hamilton - Jacobi theory

Different formulation of mechanics

Fix initial condition $x_0 \in X$, set $S(x,t)$ as action for physical path connecting (x_0, t_0) to (x,t).

Say we move the final endpoint to $x + \Delta x$
to first order curve γ moves to $\gamma + \delta\gamma$
then action changes:

$$\delta S = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial x} \cdot \delta\gamma + \frac{\partial L}{\partial v} \delta\dot{\gamma} \right) ds$$

$$= \int_{t_0}^{t_1} \underbrace{\left(\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial v} \right)}_{\text{along } \gamma} \delta\gamma \, ds + \left[\frac{\partial L}{\partial v} \delta\gamma \right]_{t_0}^{t_1}$$

$$= 0 + \frac{\partial L}{\partial v} \cdot \Delta x$$

E-L equations hold along physical path

$\delta\gamma(t_1) = \Delta x$

$$\Rightarrow \boxed{\frac{\partial S}{\partial x} = p}$$

Diff wrt t. Along physical path, $\frac{dS}{dt} = L$

also by chain rule, $\frac{dS}{dt} = \frac{\partial S}{\partial t} + \frac{\partial S}{\partial x} \cdot \frac{dx}{dt}$

$$= \frac{\partial S}{\partial t} + p \cdot v$$

$$\Rightarrow \frac{\partial S}{\partial t} = L - pv = -H$$

Conclusion: $\frac{\partial S}{\partial t} + H(x; p; t) = 0$

$$\Leftrightarrow \boxed{\frac{\partial S}{\partial t} + H\left(x; \frac{\partial S}{\partial x}; t\right) = 0}$$

Hamilton-Jacobi equation.

New pov: H is given. Then this is a nonlinear PDE for S. Solving for S, integrates equations of motion

Example: Harmonic oscillator.

$$H = \frac{1}{2m} p^2 + \frac{1}{2} k x^2$$

Get PDE $\quad \frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + \frac{1}{2} k x^2 = 0$

Can solve this as $S = W(x) + f(t)$ (Ansatz)

then $\frac{1}{2m}\left(\frac{dW}{dx}\right)^2 + \frac{1}{2} k x^2 = -\frac{df}{dt}$

has to be a constant α

Plug in $\quad S = W(x) - \alpha t \quad$ get ODE

$$\frac{1}{2m}(W')^2 + \frac{1}{2} k x^2 = \alpha$$

$$\Rightarrow W' = \sqrt{2m\alpha - km\, x^2} \Rightarrow W = \int \sqrt{2m\alpha - km\, x^2}\, dx$$

take α as our co-ord

Then corresponding momentum is $\beta = \frac{\partial S}{\partial \alpha} = \frac{\partial W}{\partial \alpha} - t$

$$\Rightarrow \beta = -t + \sqrt{\frac{m}{2\alpha}} \int \frac{dx}{\sqrt{1-\frac{kx^2}{2\alpha}}} = -t + \frac{1}{\omega} \arcsin\left(\sqrt{\frac{k}{2\alpha}}\, x\right)$$

$(\omega = \sqrt{k/m})$ $\quad\Rightarrow\quad x = \sqrt{\frac{2\alpha}{k}} \sin(\omega t + \omega\beta)$

$$= \sqrt{\frac{2\alpha}{k}} \sin(\omega t + \phi)$$

$$p = \frac{\partial S}{\partial x} = \frac{dW}{dx} = \sqrt{2m\alpha - mkx^2} = \sqrt{2m\alpha}\cos(\omega t + \phi)$$

$$\Rightarrow \quad \frac{1}{2}\frac{p^2}{m} + \frac{1}{2}kx^2 = \alpha$$ so α = total energy

Similarly $\tan\phi = \sqrt{km}\,\frac{x_0}{p_0}$

map $(x_0, p_0) \mapsto (x_t, p_t)$ is a symplectomorphism
ie. a time-dependent canonical transformation.