Math 428, Lecture 20

Last time: Intro to QM

Today: Mathematical quantization

(1) Fourier transform

(2) Pseudodifferential operators

(3) Microlocal calculus

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① Review of Fourier analysis

1(a) on $S^1 = \mathbb{R}/\mathbb{Z}$. Set $e(z) = \exp(2\pi i z)$

For $k \in \mathbb{Z}$, $e_k(x) \overset{\text{def}}{=} e(kx) \in L^2(S^1)$

$\{e_k\}_{k\in\mathbb{Z}} \subset L^2(S^1)$ orthonormal

$A = \operatorname{Span}_{\mathbb{C}}\{e_k\}_{k\in\mathbb{Z}} \subset C(S^1)$ subalgebra ($e_k e_\ell = e_{k+\ell}$)

contains $e_0 \equiv 1$

e_1 separates points

$\overline{e_k} = e_{-k}$.

$\Rightarrow$ (Stone-Weierstrass) A is dense in $C(S^1)$

$\Rightarrow$ $A^\perp = C(S^1)^\perp = L^2(S^1)^\perp = \{0\}$ so $\{e_k\}$ is an o.n.b.

$\Rightarrow$ $f \in L^2(S^1)$, $\hat{f}(k) = \langle e_k, f\rangle$ then $f = \sum_k \hat{f}(k) e_k$ in L^2,

$\|f\|_{L^2}^2 = \sum_k |\hat{f}(k)|^2$ (Parseval's identity)

<u>Check</u> (Integration by parts) if $f \in C^r(S^1)$ then

$$|\hat{f}(k)| \ll \langle k \rangle^{-r} \qquad \langle k \rangle = \sqrt{1+|k|^2}$$

$\Rightarrow$ (e_k.) if $f \in C^2$ then $\sum_k \hat{f}(k) e_k(x)$ converges abs to f

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Ex: $\Lambda < \mathbb{R}^n$ discrete subgp $\Leftrightarrow$ $\Lambda = \bigoplus_{i=1}^{r} \mathbb{Z} v_i$, $\{v_i\}_{i=1}^r \subset \mathbb{R}^n$ indep $/\mathbb{R}$

$\mathbb{R}^n/\Lambda$ cpt $\Leftrightarrow$ $r = n$ $\Leftrightarrow$ $\{v_i\}_{i=1}^n \subset \mathbb{R}^n$ basis

call Λ a lattice.

Entire theory works on $\mathbb{R}^n/\Lambda$: want $\int_{\mathbb{R}^n/\Lambda} f(x) e(-kx)\,dx$ to make sense so need $e(-k\lambda) = 1$ for all $\lambda \in \Lambda$

$k \in (\mathbb{R}^n)^*$

$\Leftrightarrow \langle k, \lambda \rangle \in \mathbb{Z}$ for all $\lambda \in \Lambda$

Dual lattice $\Lambda^* = \{ k \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}) \mid k|_\Lambda \in \mathrm{Hom}(\Lambda, \mathbb{Z}) \}$

$= \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$.

Then $\hat{f} : \Lambda^* \to \mathbb{C}$, $\hat{f}(k) = \frac{1}{\mathrm{vol}(\mathbb{R}^n/\Lambda)} \langle e_k, f \rangle$, $e_k(x) = e(kx)$

Then $f = \sum_{k \in \Lambda^*} \hat{f}(k) e_k$ in L^2, $f(x) = \sum_{k \in \Lambda^*} \hat{f}(k) e_k(x)$

if enough decay (e.g. $f \in C^{n+1}(\mathbb{R}^n)$).

On $\mathbb{R}^n$: if $f \in L^1(\mathbb{R}^n)$ set $\hat{f}(k) = \int_{\mathbb{R}^n} f(x) e(-kx) dx$
for $k \in (\mathbb{R}^n)^*$.

<u>Def:</u> $f \in C^\infty(\mathbb{R}^n)$ is of Schwartz class if for each α, r

$$|\partial^\alpha f(x)| \leq C_\alpha \cdot \langle x \rangle^{-r}.$$

Write $\mathcal{S}(\mathbb{R}^n)$ for the vsp.

<u>Ex:</u> let $f \in \mathcal{S}(\mathbb{R}^n)$. Then: $\qquad D^j = \partial_j = \frac{\partial}{\partial x^j}$

(1) $\widehat{\partial_j f}(k) = \int_{\mathbb{R}^n} \partial_j f(x) e(-kx) dx = (2\pi i k_j) \hat{f}(k)$

$\Rightarrow D^\alpha = (2\pi i)^{|\alpha|} \cdot k^\alpha \hat{f}(k)$

(2) $(\partial_j \hat{f})(k) = \widehat{(-2\pi i x_j) f}(k)$

$\Rightarrow \partial_k^\alpha \hat{f} = (-2\pi i)^{|\alpha|} \widehat{x^\alpha f}(k) \ll \langle k \rangle^{-r}$
decays

Get: $\hat{f} \in \mathcal{S}(\mathbb{R}^n)$

Set $\Phi_\Lambda(x) = \sum_\lambda f(x+\lambda)$ $\quad$ • $\Phi_\Lambda : \mathbb{R}^n/\Lambda \to \mathbb{C}$

$\Phi_\Lambda \in C^\infty(\mathbb{R}^n/\Lambda)$

check $\hat{\Phi}_\Lambda(k) = \frac{1}{\mathrm{vol}(\mathbb{R}^n/\Lambda)} \hat{f}(k)$ for $k \in \Lambda^* \subset (\mathbb{R}^n)^*$.

By Fourier inversion $\Phi_\Lambda(x) = \frac{1}{\mathrm{vol}(\mathbb{R}^n/\Lambda)} \sum_{k \in \Lambda^*} \hat{f}(k) e_k(x)$

Consider $\Phi_{\delta\Lambda}(x) = \frac{1}{\underbrace{\delta^n \mathrm{vol}(\mathbb{R}^n/\Lambda)}_{\mathrm{vol}((\mathbb{R}^n)^*/\delta^{-1}\Lambda^*)}} \sum_{k \in \delta^{-1}\Lambda^*} \hat{f}(k) e_k(x)$

$\delta \to \infty$: $\Phi_{\delta\Lambda}(x) \to \Phi(x)$ $\qquad$ $\delta \to \infty$: $\to \int_{(\mathbb{R}^n)^*} \hat{f}(k) e_k(x)\, dk$

Similarly: $\sum_{k \in \delta^{-1}\Lambda^*} |\hat{\Phi}_{\delta\Lambda}(k)|^2 = \frac{1}{\delta^2 \mathrm{vol}(\mathbb{R}^n/\Lambda)} \int_{\mathbb{R}^n/\delta\Lambda} |\Phi_{\delta\Lambda}(x)|^2 dx$

$\Rightarrow \mathrm{vol}((\mathbb{R}^n)^*/\delta^{-1}\Lambda^*) \sum_{k \in \delta^{-1}\Lambda^*} |\hat{f}(k)|^2 = \sum_{\lambda, \lambda' \in \Lambda} \int \overline{f(x+\delta\lambda)} f(x+\delta\lambda')\, dx$

take $\lambda \to \infty$ $\qquad \downarrow \int_{(\mathbb{R}^n)^*} |\hat{f}(k)|^2 dk \qquad \downarrow \int_{\mathbb{R}^n} |f(x)|^2 dx$

Plancherel's Identity.

Cor: $f \mapsto \hat{f}$ L^2 isometry $\Rightarrow$ extend to isom $L^2 \to L^2$.

Semiclassical FT: For $p \in (\mathbb{R}^n)^*$ set

$$\tilde{f}(p) = \hat{f}\left(\frac{p}{h}\right) = \int_{\mathbb{R}^n} f(x)\, e\left(-\frac{px}{h}\right) dx = \int_{\mathbb{R}^n} f(x) \exp\left(-\frac{px}{\hbar}\right) dx.$$

Then $$f(x) = h^{-n} \int \tilde{f}(p)\, e\left(\frac{px}{h}\right) dp$$

$$\|f\|_{L^2}^2 = h^{-n} \int |\tilde{f}(p)|^2 dp = h^{-n} \|\tilde{f}\|_{L^2}^2$$

② **Quantization** = ΨDO (Kohn-Nirenberg)

Want to quantize obervable x by M_x
$(M_x f)(x) = x f(x)$

more generally if $a = a_1(x)$ want $(M_{a_1} f)(x) = a_1(x) f(x)$

Wanted to quantize p_j by $-i\hbar \partial_j$

Know: $\widehat{\partial_j f}(k) = 2\pi i k_j \hat{f}(k)$

$\Rightarrow -i\hbar\partial_j$ is a Fourier mult: $\tilde{f} \mapsto p_j \tilde{f}$

Want to quantize $a_1(x)\, a_2(p)$

One choice (apply $a_2(p)$ first) gives:

$$Op_h^{KN}(f) = h^{-n} a_1(x) \int a_2(p)\, \hat{f}(p)\, e\left(\frac{px}{h}\right) dp$$

$$= h^{-n} \int a(x,p)\, \hat{f}(p)\, e\left(\frac{px}{h}\right) dp$$

$$= h^{-n} \iint a(x,p)\, f(y)\, e\left(\frac{p(x-y)}{h}\right) dy\, dp$$

translation invariant

phase space integral

More generally set $(t \in [0,1])$

$$\left(Op_h^t(a) f\right)(x) = h^{-n} \iint_M a\left((1-t)x + ty, p\right) f(y)\, e\left(\frac{p(x-y)}{h}\right) dy\, dp$$

(1) $Op_h^0 = Op_h^{KN}$

(2) Op_h^1 = "adjoint K-N quantization" has

$$Op_h^1(a_1(x)\, a_2(p)) = Op_h(a_2)\, Op_h(a_1)$$

(3) $t = \frac{1}{2}$ is Weyl symmetrization

Ex: $Op_h^W(xp) = \dfrac{\hat{x}\hat{p} + \hat{p}\hat{x}}{2}$.

Notes

$$\langle Op_h^W(a)f, g\rangle = h^{-n}\iiint \overline{a\left(\tfrac{x+y}{2}, p\right)\,\overline{f(y)}}\, g(x)\, e\left(\tfrac{p(y-x)}{h}\right) dx\,dy\,dp$$

$$= \langle f, Op_h^W(\bar{a})g\rangle$$

Ex: $(Op_h^t(a))^\dagger = Op_h^{1-t}(\bar{a})$.

Observe: If $a(x,p) = a_2(p)$ then $Op_h^t(a)$ = Fourier mult by $a_2(p)$

If $a(x,p) = \sum_{|\alpha|\le m} a_\alpha(x)\, p^\alpha$

then $Op_h^{t=0}(a)$ = diff operator $\sum_\alpha a_\alpha(x)\left((-ih)^{|\alpha|}\partial^\alpha\right)$

Theorem: $a \in \mathcal{S}(\mathbb{R}^{2n})$ then $Op_h^t(a)$ makes sense on very general f. Gives cts map $\mathcal{S}(\mathbb{R}^n)' \to \mathcal{S}(\mathbb{R}^n)$

Pf: $(Op_h^t(a)f)(x) = \int_{\mathbb{R}^n} K(x,y)\, f(y)\, dy$

$$K(x,y) = h^{-n}\int_{(\mathbb{R}^n)^*} a([x,y]_t, p)\, e\left(\tfrac{p(x-y)}{h}\right) dp$$

partial FT

So same as before $K(x,y) \in \mathcal{S}(M)$

We see $Op_h^t(a)$ bdd on any reasonable fcn space (e.g. $L^2 \to L^2$, $\mathcal{S} \to \mathcal{S}$, $L^p \to L^p$).

Prop: $Op_h^t(a_1(x))$ is the multiplier M_{a_1}

(know this if $t=0$)

Pf: $\frac{d}{dt}\left(Op_h^t(a_1(x))f\right) =$

$$h^{-n}\iint \langle \partial_x a([x,y]_t, p), y-x\rangle f(y) e\left(\frac{p(x-y)}{h}\right) dy\,dp$$

$$= \frac{h}{2\pi i h^n}\iint \langle \partial_x a_1([x,y]_t), \partial_p e\left(\frac{p(x-y)}{h}\right)\rangle f(y)\,dy\,dp$$

$$= -\frac{h}{2\pi i h^n}\int_{(\mathbb{R}^n)^*} dp\ \mathrm{div}_p\left[e\left(\frac{px}{h}\right)\cdot \widehat{\left[\partial_x a_1([x,y]_t) f(y)\right]}(p)\right] = 0$$

$\blacksquare$