Math 428, lecture 21

Last time: Quantization

Fourier transform: $e(z) = e^{2\pi i z}$, if $f : \mathbb{R}^n \to \mathbb{C}$, $p \in (\mathbb{R}^n)^*$

$$\hat{f}(p) = \int_{\mathbb{R}^n} f(x)\, e\left(-\frac{px}{h}\right) dx$$

$\exp\left(-\frac{ipx}{\hbar}\right)$

Then $f(x) = h^{-n} \int_{(\mathbb{R}^n)^*} \hat{f}(p)\, e\left(\frac{px}{h}\right) dp$

$X = \mathbb{E}^n$

$M = T^*X$
$= \mathbb{R}^n \times (\mathbb{R}^n)^*$

Quantization: for $a : M \to \mathbb{C}$, $f : X \to \mathbb{C}$

$$\left(Op_h^t(a) f\right)(x) = h^{-n} \iint a\left((1-t)x + ty, p\right) f(y)\, e\left(\frac{p(x-y)}{h}\right) dp\, dy$$

(Mostly think of $Op_h^W(a) = Op_h^{\frac{1}{2}}(a)$)

Interpretation: If $a \in \mathcal{S}(M)$, can take $f \in \mathcal{S}(\mathbb{R}^n)'$
get $Op_h^W(a) f \in \mathcal{S}(M)$

Lemma: $\left[Op_h(a)\left(e\left(\frac{p\cdot}{h}\right)\right)\right](x) = a(x,p)\, e\left(\frac{px}{h}\right)$
$a \in \mathcal{S}(M)$

Pf:

$$e\left(-\frac{px}{h}\right)\left[\mathrm{Op}_h(a)\, e\left(\frac{p\,\cdot}{h}\right)\right](x) =$$

$$h^{-n}\iint \underbrace{a(x,p')}_{\text{symbol}}\, \underbrace{e\left(\frac{py}{h}\right)}_{f(y)} e\left(\frac{-px}{h}\right) e\left(\frac{p'(x-y)}{h}\right) dy\,dp'$$

$$= h^{-n}\int dp'\, a(x,p')\, e\left(\frac{(p'-p)x}{h}\right)\int dy\, e\left(\frac{(p-p')y}{h}\right)$$

$$= \int dp'\, a(x,p')\, e\left(\frac{(p'-p)x}{h}\right)\delta(p-p') = a(x,p) \qquad \blacksquare$$

Prop: $a, b \in S(M)$. Then $\mathrm{Op}_h^W(a)\,\mathrm{Op}_h^W(b) = \mathrm{Op}_h^W(a\#b)$

where $$a\#b = \left[\sum_{k=0}^{N} \frac{i^k h^k}{k!} A(D)^k a(x,p)\, b(x',p') + O(h^{N+1})\right]_{\substack{x=x' \\ y=y'}}$$

$$A(D) = w\left(\begin{pmatrix}\partial_x \\ \partial_p\end{pmatrix}, \begin{pmatrix}\partial_{x'} \\ \partial_{p'}\end{pmatrix}\right)$$

$$\Rightarrow \quad a\#b = ab + \frac{h}{2i}\{a,b\} + O(h^2)$$

$$\Rightarrow \left[\mathrm{Op}_h^W(a), \mathrm{Op}_h^W(b)\right] = ih\, \mathrm{Op}_h^W(\{a,b\}) + O(h^3)$$

(h^3 special to Op_h^W, in general $O(h^2)$)

Today: More general observables

Ex: for $f \in \mathcal{S}(\mathbb{R}^n)$, can take a of moderate growth

$$\partial_x^\alpha \partial_p^\beta a(x,p) \ll_{a,\alpha,\beta} \langle x \rangle^{O_{a,\alpha}(1)+r|\beta|} \langle p \rangle^{O_{a,\alpha,\beta}(1)}$$

Won't go there. We will allow a to depend on h

Def: Write $a \in S^m(\mathbb{R}^n)$ say "a is a symbol of order m"

if $$\partial_x^\alpha \partial_p^\beta a(x,p) \ll_{a,\alpha,\beta} \langle p \rangle^{m-|\beta|}$$ (e.g. poly in p)

Write $a \in S^m_\delta$ if

$$\partial_x^\alpha \partial_p^\beta a(x,p) \ll_{a,\alpha,\beta} h^{-\delta(|\alpha+\beta|)} \langle p \rangle^{m-|\beta|}$$

In both cases need implied constants to be uniform in $0 < h \leq h_0$

Ex: $a(x,p) = \sum_{|\beta| \leq m} c_\beta(x) p^\beta$ $\quad$ $c_\beta(x)$ & all its derivatives bounded

Quite often h dependence takes form

$$a \sim \sum_{j=0}^{\infty} a_j h^j \quad \text{i.e.} \quad a - \sum_{j=0}^{N-1} a_j h^j = O(h^N)$$

in relevant function space

Thm: (Borel) Given $a_j \in S^m$ have $a \in S^m$ s.t. $a \sim \sum_{j=0}^{\infty} a_j h^j$

Prop: If $a \in S^m_\delta$, $b \in S^{m'}_\delta$ then $a \# b \in S^{m+m'}_\delta$.

Thm: $Op_h^W(a)\, Op_h^W(a) = Op_h^W(a \# b)$ in $S^m_\delta, S^{m'}_\delta$.

Thm: If $a \in S^0$, then $Op_h^W(a)$ is bounded on $L^p(\mathbb{R}^n)$ for all $1<p<\infty$, uniformly in h. (mild h-dependence if $a \in S^0_\delta$).

Pf: Littlewood-Paley decomposition.

Cor: If $a, b \in S^0_\delta$, $Op_h^W(a)\, Op_h^W(b) = Op_h^W(ab) + O_{L^2 \to L^2}(h^{1-2\delta})$

(theory only makes sense if $\delta \le \frac{1}{2}$)

Cor: If a, b have disjoint supports $Op_h^W(a)\, Op_h^W(b) = O(h^\infty)$

Cor: let $\psi \in L^2(X)$, set $\psi(t) = U(t)\psi$ where

$$i\hbar \frac{dU(t)}{dt} = \hat{H}\, U(t) \qquad \hat{H} = Op_\hbar^w(H).$$

let $A = Op_\hbar^w(a)$. Then $(a, H \in S^0)$

$$\frac{d}{dt}\langle \psi(t) | A | \psi(t) \rangle = \frac{1}{i\hbar}\langle \psi(t) | [\hat{H}, A] | \psi(t) \rangle$$

$\sum_{\lambda \in \sigma(A)} \|P_{V_\lambda} \psi(t)\|^2 \cdot \lambda$ = expected value of measuring A at state $\psi(t)$

$$= \langle \psi(t) | Op_\hbar^w(\{H, A\}) | \psi(t) \rangle + O(\hbar)$$

$$= \langle \psi(t) | Op_\hbar^w(X_H \cdot a) | \psi(t) \rangle + O(\hbar)$$

(Ehrenfest's Theorem).

Pf: $\frac{d}{dt}\langle U(t)\psi, AU(t)\psi \rangle = \frac{d}{dt}\langle \psi, U(t)^{-1} A U(t)\psi \rangle$

and $\frac{d}{dt} U(t)^{-1} A U(t) = \frac{1}{i\hbar}\left(-U(t)^{-1}\hat{H} A U(t) + U(t)^{-1} A \hat{H} U(t)\right).$

Egorov's Theorem

(1) Time-dependent Hamiltonian, fixed interval

$H \in C^\infty(M_t \times \mathbb{R}')$, suppose for all t, $\mathrm{supp}(H(\cdot,t))$ is contained in a fixed bdd set.
$\Rightarrow H(t) \in C_c^\infty(M) \subset \mathcal{S}(M)$.

let Φ_t = Hamiltonian flow, $\hat{H}(t) = Op_h^w(H(t))$, $U(t)$ defined by

$$ih \frac{dU}{dt}(t) = U(t)\hat{H}(t)$$

(Thm: this has a unique solution, $U(t)^{-1} = U(t)^*$)

<u>Thm</u>: (Egorov) let $a \in S^m$, $T \geq 0$. Then on $[0,T]$

$$A = Op_h^w(a)$$

$$B(t) = U(t)^{-1} A\, U(t) = Op_h^w(b_t)$$

with $b_t = a \circ \Phi_t + O_{S^m}(h)$

<u>Pf:</u> $a \circ \Phi_t \in S^m$: the vector fields $X_{H(t)}$ have compact support.

Set $B_0(t) = Op_h^w(a \circ \Phi_t)$ Then

$$ih \frac{d}{dt} B_0(t) = ih\, Op_h^w(\{H(t), a \circ \Phi_t\})$$
$$= [\hat{H}(t), B_0(t)] + E(t)$$

$E(t) = Op_h^w(e(t))$, $e(t) \in h^2 S(M)$

supp $\uparrow$ $H(t) \# a \circ \Phi_t \subseteq$ supp$(H(t))$.

$\|E(t)\|_{L^2} = O(h^2)$

$\Rightarrow ih \frac{d}{dt}\left(U(t) B_0(t) U(t)^{-1}\right) = U(t) \hat{H}(t) \cancel{B_0(t)} U(t)^{-1}$

$U(t) \cancel{[\hat{H}, B_0(t)]} U(t)^{-1} + U(t) E(t) U(t)^{-1}$

$\cancel{- U(t) B_0(t) \hat{H}} U(t)^{-1}$

$= U(t) E(t) U(t)^{-1} = O_{L^2 \to L^2}(h^2)$

So $U(t) B_0(t) U(t)^{-1} - A = \frac{1}{ih} \int_0^t U(s) E(s) U(s)^{-1} ds$

$= O(h)$

$\Rightarrow$

$B(t) - B_0(t) = \frac{i}{h} U(t)^{-1} \int_0^t U(s) E(s) U(s)^{-1} ds \; U(t)$

$U(t)^{-1} A U(t)$

Technical part: RHS is $Op(e')$

e' size $O_{S^m}(h)$.

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① If $a \sim \sum_{j=0}^{\infty} a_j h^j$ set $b_t = a_0 \circ \Phi_t + \sum_{j=1}^{\infty} b_j h^j$

PDE for b_j harder

② Proof also works if $H, a \in S^0$.