Math 428, lecture 22

① WKB ② QUE

Last time: Egorov's Theorem : $U(t) = \exp\left(\frac{\hat{H}t}{ih}\right)$
$\hat{H} = Op_h^W(H)$

then $U(t)^{-1} Op_h^W(a)\, U(t) = Op_h^W(b_t)$

$b_t = a \circ \Phi_t + O(h^\infty)$

Φ_t = Hamiltonian flow of H,
$a \in S_\delta^m$

for $0 \leq t \leq \frac{C \cdot |\log h|}{\gamma}$ γ = Lyapunov exponent

"Ehrenfest time"

Different connection to CM: WKB

Wentzel-Kramers-Brillouin.

Example: $H = \frac{p^2}{2m} + V(x)$, $Op_h^W(H) = -\frac{h^2}{2m}\Delta + V$

Schrödinger Equation reads $ih\frac{\partial}{\partial t}\psi = \hat{H}\psi$

Try ansatz $\Psi(x,t) = \exp\left(\frac{iS(x,t)}{\hbar}\right) = e\left(\frac{iS}{\hbar}\right).$

Get $$-\frac{\partial S}{\partial t}\psi = \left[\frac{1}{2m}|\nabla S|^2 + \frac{i\hbar}{2m}\Delta S + V\right]\psi$$

$$\Leftrightarrow \qquad -\frac{\partial S}{\partial t} = \frac{1}{2m}|\nabla S|^2 + V + \frac{i\hbar}{2m}\Delta S$$

Natural to try $S \sim \sum_{j=0}^{\infty} \hbar^j S_j$

$$\Rightarrow \quad -\frac{\partial S_0}{\partial t} = \frac{1}{2m}|\nabla S_0|^2 + V = H(x, \nabla S_0)$$

Hamilton-Jacobi Equation! $\Rightarrow$ take $S_0 =$ classical action

Know $S_0(x,t) = W(x) - \epsilon t$

with

$$|\nabla W|^2 = 2m(\epsilon - V) \qquad \text{Eikonal equation}$$

(eg. in 1d $W(x) = \int^x \sqrt{2m(\epsilon - V)} = \int^x p(\xi)\,d\xi$

$p(x) = \frac{\partial S}{\partial x} =$ momentum system has at position x if Energy $= \epsilon$.)

S_j satisfy transport equations can be estimated recursively in some situations.

Remarks

① Ψ is defined even when $V(x) > E$.

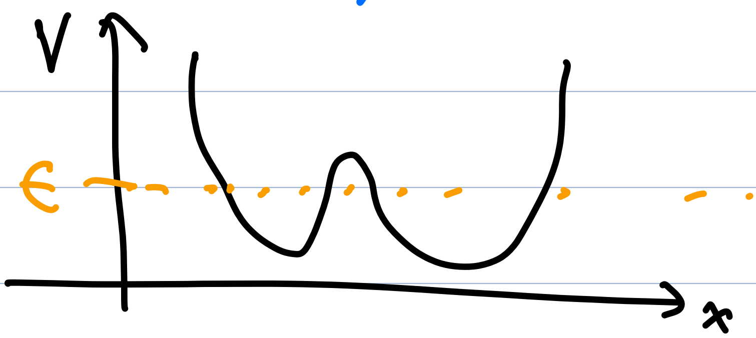
(a) in region $\{V(x) < E\}$ ("classically permitted")

W is real, S real, $e^{\frac{iS}{\hbar}}$ oscillates

(b) in region $\{V(x) > E\}$ ("classically forbidden")

$\nabla W = \sqrt{2m(E-V)}$ is imaginary, $e^{\frac{iS}{\hbar}}$ varies exponentially

(typically exponential decay) "tunneling"



② $S_0 = W(x) - Et$ gives $\Psi \approx e^{\frac{iW(x)}{\hbar}} e^{\frac{Et}{i\hbar}}$

(recovered time-indep Schrödinger equation)

③ Can apply this to general (say linear) PDE

Makes sense $Op(H)$ any H.

Look at ODE $\hat{H} = \sum_{j=0}^{k} a_j(x) \hbar^j \frac{d^j}{dx^j}$, $\hat{H}\Psi = 0$

try $\Psi = e^{\frac{iS(x)}{\hbar}}$, $S \sim \sum_m \hbar^m S_m(x)$

(a) Wave equation $\frac{\partial^2}{\partial t^2}\psi = c^2 \Delta \psi$

Get "original" Eikonal equation

approximation : "geometric optics".

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(2) "Quantum Unique Ergodicity":

If $\psi \in L^2(X)$ define the associated Wigner measure as

$$\mu_\psi(a) = \langle \psi | Op_h(a) | \psi \rangle = \langle \psi, Op_h(a) \psi \rangle$$

μ_ψ is a linear functional $S^m_\delta \to \mathbb{C}$.

<u>Ex</u>: Since $Op(a)$ bdd in L^2, μ_ψ is a distribution. (have bound

$$|\mu_\psi(a)| \ll_\psi \|a\|_{H^k} \leftarrow \text{Sobolev norm}$$

(if a is small, want $\mu_\psi(a)$ to be small)

<u>Physics</u>: For the Weyl quantization scheme call this "Wigner function".

Know: $Op_h(a)\,Op_h(b) = Op_h(ab) + O_{L^2\to L^2}(h)$

Suppose $a \in S^0$, $a \geq 0$, then $\sqrt{a}$ is a symbol (at least if $a(x_0,p_0)>0$ take smooth cutoff χ near (x_0,p_0) s.t. $a>0$ on supp of χ)

Then $\sqrt{a}\cdot\chi$ is a symbol, $(\sqrt{a}\,\chi)^2 = a\,\chi^2$

$$Op_h(a\chi^2) = Op_h^W(\sqrt{a}\,\chi)^2 + O(\hbar)$$

$$\Downarrow$$

$$\mu_{\psi_h}(a\chi^2) = \| Op_h^W(\sqrt{a}\,\chi)\psi_h\|^2 + O(\hbar)$$

Corollary: Any weak-* limit $\mu_0 = \lim_{h\to 0} \mu_{\psi_h}$

has property $\mu_0(a) \geq 0$ if $a \geq 0$

Also $\mu_0(\mathbb{1}) = 1$

So μ_0 is a probability measure (if X is cpt)

Let $E_h = E + O(h)$, Suppose $\hat{H}\psi_h = E_h\psi_h$

Example: $H = \frac{p^2}{2m} \rightsquigarrow \hat{H} = -\frac{\hbar^2}{2m}\Delta$

$$-\frac{\hbar^2}{2m}\Delta\psi = E_h \psi \iff -\Delta\psi = \frac{2m E_h}{\hbar^2}\psi$$

define $\hbar = \sqrt{\frac{2mE}{\lambda}}$

semiclassical limit $\hbar \to 0$

$\shortparallel$

high frequency limit $\lambda \to \infty$

Warning: if $H = \frac{p^2}{2m} + V$, $\hat{H} = -\frac{\hbar^2 \Delta}{2m} + V$

limits $\hbar \to 0$, E fixed

$\hbar$ fixed, $\lambda \to \infty$ different!

Study $\hat{H}\psi_h = E_h \psi_h$, $\mu_h(a) = \langle \psi_h | Op_h(a) | \psi_h \rangle$.

fixed t

Egorov: $U(t)^{-1} Op_h(a) U(t) = O_h(a \circ \Phi_t) + O_t(h^\infty)$

$$\Rightarrow \mu_h(a \circ \Phi_t) = \langle \psi_h | U(t)^{-1} Op_h(a) U(t) | \psi_h \rangle$$
$$= \langle U(t)\psi_h | Op_h(a) | U(t)\psi_h \rangle + O(h^\infty) =$$

$\left[\text{But: } U(t)\psi_h = e\left(\frac{E_h t}{ih}\right) \cdot \psi_h\right]$

$$= \overline{e\left(\frac{E_h t}{ih}\right)}\, e\left(\frac{E_h t}{ih}\right) \langle \psi_h | Op_h(a) | \psi_h \rangle + O(h^\infty)$$

$$= \mu_h(a) + O(h^\infty) \qquad \Rightarrow \mu_0(a \circ \Phi_t) = \mu_0(a)$$

(Aside: don't need $\hat{H}\psi_h = E_h \psi$ exactly,
enough $\|(\hat{H} - E_h)\psi_h\| \xrightarrow[h \to 0]{} 0$
("quasimodes"))

Def: Call μ_0 "quantum limit".
Problem: Which Φ_t-inv't prob measures on M arise this way?

① If system is quantum completely integrable: have commuting observables $\{J_j\}$ s.t. $Op_h(J_j)$ commute

then expect to get any measure

② If system is very chaotic, expect only Liouville measure

① $\mathbb{R}^n/\mathbb{Z}^n$ $\quad e(\underline{k}\cdot\underline{x}) \quad \underline{k} \in \mathbb{Z}^n$

$\lambda = |\underline{k}|^2$