EQUIVALENCE RELATIONS
(NOTES FOR STUDENTS)

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1. RELATIONS

1.1. List of examples.

- *Equality* of real numbers: for some $x, y \in \mathbb{R}$ we have $x = y$. For other pairs this isn't true.
- The *order relation* on R, usually denoted $x \leq y$.
- *Set membership*: for some sets x, y we have $x \in y$.
- *Divisibility* in $\mathbb{Z}$: 3 divides 12 but 5 doesn't divide 12 (in notation, $3|12$ but $5 \nmid 12$).
- *Divisibility* in $\mathbb{Z}[x]$: $(x+1)|(x^5+3x^2-2)$ but $x^2 \nmid (x^5+3x^2-2)$.

1.2. Relations.

1.2.1. *Informal discussion.* We fix a set X (the "universe"). Informally, a *relation* on X is a property of pairs of elements from X. For examples, "equality" is a property of pairs of real numbers (some pairs consist to two equal numbers, some don't). Similarly, "less than" is a property of pairs of real numbers (we usually call that the "order relation on $\mathbb{R}$"). On the other hand, $f(x,y) = x+y$ is not a relation – it is a *function* of the two variables x, y. "Is even" is not a relation on $\mathbb{Z}$ because it is a property of individual integers, not pairs, but "a divides b" is a relation on the integers. If R is a relation we usually write xRy to say that x is related to y, and $x \not R y$ to say the opposite. Examples of this notation include:

- Equality:$x = y, x \neq y$
- Order: $x < y$ (but we usually write $y \leq x$ rather than $x \not< y$).
- Divisibility: $a|b$, $a \nmid b$

1.2.2. *Formalization.* We can encode relations using set theory. For this write $X \times X$ for *Cartesian product,* that is the set of *pairs* $\{(x,y) \mid x, y \in X\}$. We can then identify a relation R with the set of pairs $\{(x,y) \mid xRy\}$. In fact, we formally take the latter point of view:

Definition 1. A *relation* on X is a subset $R \subset X \times X$. Write xRy for $(x,y) \in R$ and $x \not R y$ for $(x,y) \notin R$.

Exercise 2. Show that the relation $\not R$ corresponds to the *complement* $X \times X \setminus R = \{p \in X \times X \mid p \notin R\}$.

Exercise 3. Let R_1, R_2 be two relations on X. Show that $R_1 \subset R_2$ iff $xR_1y \Rightarrow xR_2y$ for all $x, y \in X$.

We can use this language for functions too.

Definition 4. A *function* is a relation $f \subset X \times X$ such that if $(x,y),(x,y') \in f$ then $y = y'$. We call $\text{Dom}(f) = \{x \in X \mid \exists y : (x,y) \in f\}$ the *domain* of f and $\text{Ran}(f) = \{y \mid \exists x : (x,y) \in f\}$ its *range*. If $x \in \text{Dom}(f)$ we write $f(x)$ for the (unique!) y such that $(x,y) \in f$.

Exercise 5. The function $f(x,y) = x + y$ has domain $\mathbb{R}^2$ and range $\mathbb{R}$. Realize it as a relation on the set $X = \mathbb{R}^2 \cup \mathbb{R}$.

1.2.3. *Restriction.* Let R be a relation on X and let $Y \subset X$. The *restriction* of R to Y, to be denoted $R\restriction_Y$, is the relation you get by only considering elements of Y. Informally, for $x, y \in Y$ we have $xR\restriction_Y y$ iff xRy. Formally, $R\restriction_Y = R \bigcap Y \times Y$.

Example 6. Let $\leq$ be the order relation of $\mathbb{R}$. Then $\leq\restriction_{\mathbb{Z}}$ is the order relation of the integers.

Exercise 7. Let R be the equality relation on X. Show that $R\restriction_Y$ is the equality relation on Y.

1.3. **Transitivity.** We fix a relation R on a set X.

Definition 8. We call the relation R *transitive* if for all $x, y, z \in X$, $xRy \wedge yRz \to xRz$.

Example 9. The order relation on the integers or the real numbers. Divisibility of integers.

Exercise 10. Let X be the set of all people. Write a sentence in words expressing the statement that the friendship relation is transitive. Is the statement true or false? Do the same with the relation "is an ancestor of".

Exercise 11. Let R be a transitive relation on X and let $Y \subset X$. Show that $R\restriction_Y$ is a transitive relation on Y.

The following exercise is very instructive but requires a bit more work than the others:

Exercise 12. Let R be a relation on a set X. Define a relation $\bar{R}$ as follow: $x\bar{R}y$ iff there is some $n \geq 1$ and a finite sequence $\{x_i\}_{i=0}^{n} \subset X$ such that $x_0 = x$, $x_n = y$ and $x_i R x_{i+1}$ for all $0 \leq i < n$. The relation $\bar{R}$ is called the *transitive closure* of R.

(1) Show that $\bar{R}$ is a relation on X such that $R \subset \bar{R}$ (cf. Exercise 3).
(2) Show that $\bar{R}$ is a transitive relation.
(3) Let R' be a transitive relation on X. Suppose $R \subset R'$. Show that $\bar{R} \subset R'$.
(4) Show that $\bar{R}$ is the smallest transitive relation on X containing R, and that

$$\bar{R} = \bigcap \left\{ R' \mid R \subset R' \subset X \times X \text{ and } R' \text{ is transitive} \right\}.$$

1.4. **Reflexivity.**

Definition 13. We say the relation R is *reflexive* if xRx for all $x \in X$.

Example 14. Equality is reflexive, but "is a sibling of" is not.

Exercise 15. Suppose R is reflexive. Show that $R\restriction_Y$ is also reflexive.

Exercise 16 (Landua's big-O notation)**.** Let $X = \{f \mid f\colon \mathbb{R}_{\geq 0} \to \mathbb{R}\}$ be the set of real-valued functions on the positive reals. We say that f *is of order* g and write $f = O(g)$ if there exist $x_0, M > 0$ such that for all $x > x_0$ we have

$$|f(x)| \leq M\,|g(x)|\,.$$

(1) Show that this defines a relation on X.
(2) Show that this relation is transitive and reflexive.
(3) Find f, g such that neither $f = O(g)$ nor $g = O(f)$ holds.
(4) Extend the relation to the set of real-valued functions f with $\mathrm{Dom}(f)$ an unbounded set of real numbers.

Remark 17. The notation $f = O(g)$ is common in analysis, algebra and theoretical computer science. In analytic number theory it is common to use Vinogradov's notation $f \ll g$ for the same relation.

1.5. **Symmetry.**

Definition 18. We say the relation R is *symmetric* if $xRy \leftrightarrow yRx$ for all $x, y \in X$.

Example 19. "Is a sibling of" is symmetric while "is a brother of" isn't (why?). Equality is symmetric, but the order relation of $\mathbb{R}$ isn't.

Exercise 20. Suppose R is symmetric. Show that $R\restriction_Y$ is also symmetric.

Definition 21. We say a reflexive relation is *anti-symmetric* if $xRy \wedge yRx \to x = y$.

Example 22. The order relation on $\mathbb{R}$: if $x \leq y$ and $y \leq x$ then $x = y$.

Definition 23. A *partial order* is a relation which is reflexive, transitive, and anti-symmetric.

Exercise 24. Let R be a partial order on X. Show that $R\restriction_Y$ is a partial order on Y.

Exercise 25.

(1) Show that the divisibility relation on set $\mathbb{Z}_{\geq 1}$ of positive integers is a partial order. Show that it has *incomparable elements* (positive integers a, b such that neither $a|b$ nor $b|a$ holds).
(2) Let $\mathcal{P}(X) = \{A \mid A \subset X\}$ be the *powerset* of X. Show that the *inclusion relation* $A \subset B$ on $\mathcal{P}(X)$ is a partial order. Show that if X has at least 2 elements then this partial order contains incomparable elements.

Definition 26. A *total* (or *linear)* order is a partial order in which every two elements are comparable: for any x, y either xRy or yRx.

Example 27. The usual order relation on $\mathbb{R}$ or $\mathbb{Z}$ is a total order.

Exercise 28. Let $\leq$ be a partial order on the finite set X.

(1) Show that $\leq$ has *maximal* elements*:* there exists $m \in X$ such that if $m \leq x$ then $m = x$.
(2) Give an example to show that the maximal element need not be unique, and need not be comparable to all other elements.
(3) Suppose now that $\leq$ is a total order. Show that the maximal element is unique, and that $x \leq m$ in fact holds for all $x \in X$.

2. EQUIVALENCE RELATIONS

Definition 29. An *equivalence relation* a relation $\equiv$ on a set X which is reflexive, symmetric and transitive.

Example 30. Equality.

Exercise 31. Decide which among reflexivity, symmetry and transitivity hold for the following relations:

(1) $=, \leq$ on $\mathbb{R}$.
(2) $xRy \leftrightarrow |x-y| \leq 1$ on $\mathbb{R}$ (so $\frac{1}{2}R\frac{3}{4}$ holds but $0R2$ doesn't).
(3) $aRb \leftrightarrow ab > 0$ on $\mathbb{Z}$ and $aRb \leftrightarrow ab \geq 0$ on $\mathbb{Z}$.

2.1. Equivalence relations.

Example 32. Let $X = \mathbb{Z}$, fix $m \geq 1$ and say $a, b \in X$ are *congruent mod m* if $m|a-b$, that is if there is $q \in \mathbb{Z}$ such that $a - b = mq$. In that case we write $a \equiv b\,(m)$.

Exercise 33. For each $1 \leq m \leq 7$ find all pairs $-5 \leq x, y \leq 10$ such that $x \equiv y\,(m)$.

Exercise 34. Show that congruence mod m is an equivalence relation (the only non-trivial part is transitivity).

Definition 35. Let $\equiv$ be an equivalence relation on X. The *equivalence class* of $x \in X$ is the set $[x]_\sim = \{y \in X \mid x \equiv y\}$ (we usually just write $[x]$ unless there is more than one equivalence relation in play).

Notation 36. For congruence mod m in $\mathbb{Z}$ we call the equivalence classes *congruence classes* and write $[a]_m$ for the congruence class mod m of $a \in \mathbb{Z}$.

Exercise 37. Find the all the congruence classes mod m where $m = 1$, $m = 2$, $m = 3$.

Exercise 38 (Equivalence classes)**.** Let $\equiv$ be an equivalence relation on a set X.

(1) Show that $x \in [x]$, that $y \in [x] \iff x \in [y]$, and that if $x \equiv y$ then $[x] = [y]$ (hint: these are equivalence to axioms)
(2) Suppose $z \in [x] \cap [y]$. Show that $[x] = [y]$.
(3) Conclude that $[x] = [y]$ iff $x \equiv y$ and that $[x] \cap [y] = \emptyset$ iff $x \not\equiv y$.
(4) Conlcude that any two equivalence classes are either equal or disjoint.

Definition 39. A *partition* of X is a set P of non-empty subsets of X such that:

(1) P *covers* X: if $x \in X$ then $x \in A$ for some $A \in P$; equivalently (check!) $X = \bigcup P$.
(2) P is *disjoint*: if $A, B \in P$ then either $A = B$ or $A \cap B = \emptyset$.

We have shown than the set of equivalence classes for an equivalence relation is a partition of X.

Exercise 40. Let P be a partition on X, and define a relation on X by $x \equiv_P y$ iff there is $A \in P$ such that $x, y \in A$.

(1) Show that $\equiv_P$ is an equivalence relation.
(2) Let $A \in P$ and let $x \in A$. Show that the equivalence class of x with respect to $\equiv_P$ is A, that is that $[x]_{\equiv_P} = A$.

2.2. **Quotients by equivalence relations.** Let $\equiv$ be an equivalence relation on the set X.

Definition 41. The *quotient* of X by $\equiv$, denoted $X/\equiv$ and called "X mod $\equiv$", is the set of equivalence classes for the relation. The *quotient map* is the map $q\colon X \to X/\equiv$ given by $q(x) = [x]$.

Example 42. $\mathbb{Z}/m\mathbb{Z} \stackrel{\text{def}}{=} \mathbb{Z}/\equiv_m = \{[0]_m, [1]_m, \ldots, [m-1]_m\}$ (exercise: show this for $m = 2$).

Definition 43. We say that $f\colon X \to Z$ *respects* the equivalence relation $\equiv$ on X if $f(x) = f(y)$ whenver $x \equiv y$. This extends naturally to multivariable functions.

Example 44. Every function respects equality.

Exercise 45. Let $+_m\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ be $+_m(a,b) = [a+b]_m$.
(1) Show that the statement "$+_m$ respects congruence mod m" is equivalent to the statement "if $a \equiv a'\,(m)$ and $b \equiv b'\,(m)$ then $a+b \equiv a'+b'\,(m)$".
(2) Prove this.

Exercise 46. Show that f respects $\equiv$ iff for each equivalence class $[x] \subset X$, the restriction $f\restriction_{[x]}$ is a constant function.

Exercise 47. Let $\bar{f}\colon X/\equiv \to Z$ be any function, and let $f = \bar{f} \circ q$ where q is the quotient map. Show that f respects the relation.

Construction 48. *Suppose $f\colon X \to Z$ respects the equivalence relation $\equiv$. Define $\bar{f}\colon X/\equiv \to Z$ by $\bar{f}([x]) = f(x)$ for any equivalence class $[x] \in X/\equiv$.*

Exercise 49 (Quotient functions)**.** (1) Show that $\bar{f}$ is *well-defined*: that for any equivalence class $A \in X/\equiv$, if we use $x \in A$ or $x' \in A$ to define $\bar{f}(A)$ we'd get the same value.
(2) Show that $f = \bar{f} \circ q$.

Conclusion 50. We have obtained a bijection between functions on X which respect $\equiv$ and functions on $X/\equiv$.

Exercise 51.
(1) Obtain a well-defined function $+_m\colon \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ such that $[a]_m +_m [b]_m = [a+b]_m$.
(2) Show that $+_m$ satisfies the usual rules of arithmetic: for all $x,y,z \in \mathbb{Z}/m\mathbb{Z}$, $(x +_m y) +_m z = x +_m (y +_m z)$, $x +_m y = y +_m x$, $x +_m [0]_m = x$, $x +_m (-x) = [0]_m$ with $-[a]_m = [-a]_m$.
(3) Similarly construct a function $\cdot_m\colon \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ such that $[a]_m \cdot_m [b]_m = [a \cdot b]_m$ and show that is satisfies $(x \cdot_m y) \cdot_m z = x \cdot_m (y \cdot_m z)$, $x \cdot_m y = y \cdot_m x$, $x \cdot_m [1]_m = x$ and $(x +_m y) \cdot_m z = x \cdot_m z +_m y \cdot_m z$.

3. APPENDIX: CHAINS UNDER INCLUSION AND ZORN'S LEMMA

3.1. Aside I: bases of vector spaces.

Definition 52. Let $(X, \leq)$ be a partially ordered set. A *chain* in X is a subset $Y \subset X$ which it totally ordered, that is such that $R \restriction_Y$ is a total order.

Exercise 53. Let V be a vector space over $\mathbb{R}$. Recall that a subset $S \subset V$ is *linearly independent* if whenver $\{\underline{v}_i\}_{i=1}^r \subset S$ are distinct and $\{a_i\}_{i=1}^r \subset \mathbb{R}$ are scalars not all of which are zero, we have $\sum_{i=1}^r a_i \underline{v}_i = \underline{0}$. Let X be the set of all linearly independent subsets of V, ordered by inclusion and let $Y \subset X$ be a chain.

(1) Show that $(X, \subset)$ is a partially ordered set.
(2) Let $\tilde{S} = \bigcup Y = \{\underline{v} \in X \mid \exists S \in Y : \underline{v} \in S\}$ and suppose that $\{\underline{v}_i\}_{i=1}^r \subset \tilde{S}$. Show that there is $S \in Y$ such that $\{\underline{v}_i\}_{i=1}^r \subset S$. (Hint: for each i there is $S_i \in Y$ such that $\underline{v}_i \in S_i$).
(3) Show that $\tilde{S} \in X$ as well.

Remark 54. We usually accept the following (a version of the *axiom of choice*).

Axiom 55 (Zorn's Lemma)**.** *Let V be a set, $X \subset \mathcal{P}(V)$ a non-empty set of subsets of V. Suppose that for any chain $Y \subset X$, the element $\bigcup Y$ also belongs to X. Then X contains elements maximal under inclusion (in the sense of Exercise 28).*

Exercise 56 (Linear Algebra)**.** Continuing Exercise 53, let X be the set of linearly independent subsets of the vector space V. Show that $S \in X$ is maximal iff S spans V. Use this to prove

Theorem 57. *Every vector space has a basis.*

3.2. Aside II: ultrafilters.

Fix a set X,

Definition 58. A *filter* on X is a non-empty set $F \subset \mathcal{P}(X)$ such that if $A, B \in F$ and $C \in X$ then $A \cap B, A \cup C \in F$ (equivalently, F is closed under intersection and under taking supersets), and such that $\emptyset \notin F$.

Exercise 59. Show that the following are filters on X:

(1) ("Dictatorship") The set $\{A \subset X \mid x_0 \in A\}$ where $x_0 \in X$ is fixed.
(2) ("co-finite filter") The set $F_{\text{cofin}} = \{A \subset X \mid X \setminus A \text{ finite}\}$, if X is infinite.

Exercise 60. Ordering the set of filters on X by inclusion, let F be a maximal filter.

(1) Show that F is an *ultrafilter*: for any $A \subset X$, either $A \in F$ or $X \setminus A \in F$ (and conversely, that every ultrafilter is a maximal filter).
(2) Show that every dictatorship is an ultrafilter.
(3) Show that every ultrafilter either contains F_{cofin} or is a dictatorship.

Exercise 61 (Ultrafilter lemma)**.** Let X be an infinite set and let $\mathcal{F}$ be the set of filters on X which contain F_{cofin}, ordered by inclusion.

(1) Let $Y \subset \mathcal{F}$ be a chain. Show that $\bigcup Y \in \mathcal{F}$.
(2) Show that $\mathcal{F}$ contains maximal elements.